Thoroughly Revised and Updated

Engineering Mathematics

For

GATE 2019
and ESE 2019 Prelims

Comprehensive Theory with Solved Examples
Including **Previous Solved Questions of**
GATE (2003-2018) and **ESE-Prelims** (2017-2018)

Note: *Syllabus of ESE Mains Electrical Engineering also covered*

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Engineering Mathematics for GATE 2019 and ESE 2019 Prelims

1st Edition	:	2009
2nd Edition	:	2010
3rd Edition	:	2011
4th Edition	:	2012
5th Edition	:	2013
6th Edition	:	2014
7th Edition	:	2015
8th Edition	:	2016
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10th Edition	**:**	**2018**

Preface

Over the period of time the GATE and ESE examination have become more challenging due to increasing number of candidates. Though every candidate has ability to succeed but competitive environment, in-depth knowledge, quality guidance and good source of study is required to achieve high level goals.

B. Singh (Ex. IES)

The new edition of **Engineering Mathematics for GATE 2019 and ESE 2019 Prelims** has been fully revised, updated and edited. The whole book has been divided into topicwise sections.

I have true desire to serve student community by way of providing good source of study and quality guidance. I hope this book will be proved an important tool to succeed in GATE and ESE examination. Any suggestions from the readers for the improvement of this book are most welcome.

B. Singh (Ex. IES)
Chairman and Managing Director
MADE EASY Group

SYLLABUS

GATE and ESE Prelims: Civil Engineering

Linear Algebra: Matrix algebra; Systems of linear equations; Eigen values and Eigen vectors.

Calculus: Functions of single variable; Limit, continuity and differentiability; Mean value theorems, local maxima and minima, Taylor and Maclaurin series; Evaluation of definite and indefinite integrals, application of definite integral to obtain area and volume; Partial derivatives; Total derivative; Gradient, Divergence and Curl, Vector identities, Directional derivatives, Line, Surface and Volume integrals, Stokes, Gauss and Green's theorems.

Ordinary Differential Equation (ODE): First order (linear and non-linear) equations; higher order linear equations with constant coefficients; Euler-Cauchy equations; Laplace transform and its application in solving linear ODEs; initial and boundary value problems.

Partial Differential Equation (PDE): Fourier series; separation of variables; solutions of one-dimensional diffusion equation; first and second order one-dimensional wave equation and two-dimensional Laplace equation.

Probability and Statistics: Definitions of probability and sampling theorems; Conditional probability; Discrete Random variables: Poisson and Binomial distributions; Continuous random variables: normal and exponential distributions; Descriptive statistics - Mean, median, mode and standard deviation; Hypothesis testing.

Numerical Methods: Accuracy and precision; error analysis. Numerical solutions of linear and non-linear algebraic equations; Least square approximation, Newton's and Lagrange polynomials, numerical differentiation, Integration by trapezoidal and Simpson's rule, single and multi-step methods for first order differential equations.

GATE and ESE Prelims: Mechanical Engineering

Linear Algebra: Matrix algebra, systems of linear equations, eigenvalues and eigenvectors.

Calculus: Functions of single variable, limit, continuity and differentiability, mean value theorems, indeterminate forms; evaluation of definite and improper integrals; double and triple integrals; partial derivatives, total derivative, Taylor series (in one and two variables), maxima and minima, Fourier series; gradient, divergence and curl, vector identities, directional derivatives, line, surface and volume integrals, applications of Gauss, Stokes and Green's theorems.

Differential equations: First order equations (linear and nonlinear); higher order linear differential equations with constant coefficients; Euler-Cauchy equation; initial and boundary value problems; Laplace transforms; solutions of heat, wave and Laplace's equations.

Complex Variables: Analytic functions; Cauchy-Riemann equations; Cauchy's integral theorem and integral formula; Taylor and Laurent series.

Probability and Statistics: Definitions of probability, sampling theorems, conditional probability; mean, median, mode and standard deviation; random variables, binomial, Poisson and normal distributions.

Numerical Methods: Numerical solutions of linear and non-linear algebraic equations; integration by trapezoidal and Simpson's rules; single and multi-step methods for differential equations..

GATE and ESE Prelims: Electrical Engineering

Linear Algebra: Matrix Algebra, Systems of linear equations, Eigenvalues, Eigenvectors.

Calculus: Mean value theorems, Theorems of integral calculus, Evaluation of definite and improper integrals, Partial Derivatives, Maxima and minima, Multiple integrals, Fourier series, Vector identities, Directional derivatives, Line integral, Surface integral, Volume integral, Stokes's theorem, Gauss's theorem, Green's theorem.

Differential equations: First order equations (linear and nonlinear), Higher order linear differential equations with constant coefficients, Method of variation of parameters, Cauchy's equation, Euler's equation, Initial and boundary value problems, Partial Differential Equations, Method of separation of variables.

Complex Variables: Analytic functions, Cauchy's integral theorem, Cauchy's integral formula, Taylor series, Laurent series, Residue theorem, Solution integrals.

Probability and Statistics: Sampling theorems, Conditional probability, Mean, Median, Mode, Standard Deviation, Random variables, Discrete and Continuous distributions, Poisson distribution, Normal distribution, Binomial distribution, Correlation analysis, Regression analysis.

Numerical Methods: Solutions of nonlinear algebraic equations, Single and Multi-step methods for differential equations.

Transform Theory: Fourier Transform, Laplace Transform, z-Transform.

Electrical Engineering ESE Mains

Matrix theory, Eigen values & Eigen vectors, system of linear equations, Numerical methods for solution of non-linear algebraic equations and differential equations, integral calculus, partial derivatives, maxima and minima, Line, Surface and Volume Integrals. Fourier series, linear, nonlinear and partial differential equations, initial and boundary value problems, complex variables, Taylor's and Laurent's series, residue theorem, probability and statistics fundamentals, Sampling theorem, random variables, Normal and Poisson distributions, correlation and regression analysis.

GATE and ESE Prelims: Electronics Engineering

Linear Algebra: Vector space, basis, linear dependence and independence, matrix algebra, eigen values and eigen vectors, rank, solution of linear equations – existence and uniqueness.

Calculus: Mean value theorems, theorems of integral calculus, evaluation of definite and improper integrals, partial derivatives, maxima and minima, multiple integrals, line, surface and volume integrals, Taylor series.

Differential equations: First order equations (linear and nonlinear), higher order linear differential equations, Cauchy's and Euler's equations, methods of solution using variation of parameters, complementary function and particular integral, partial differential equations, variable separable method, initial and boundary value problems.

Vector Analysis: Vectors in plane and space, vector operations, gradient, divergence and curl, Gauss's, Green's and Stoke's theorems.

Complex Analysis: Analytic functions, Cauchy's integral theorem, Cauchy's integral formula; Taylor's and Laurent's series, residue theorem.

Numerical Methods: Solution of nonlinear equations, single and multi-step methods for differential equations, convergence criteria.

Probability and Statistics: Mean, median, mode and standard deviation; combinatorial probability, probability distribution functions - binomial, Poisson, exponential and normal; Joint and conditional probability; Correlation and regression analysis.

GATE: Instrumentation Engineering

Linear Algebra : Matrix algebra, systems of linear equations, Eigen values and Eigen vectors.

Calculus : Mean value theorems, theorems of integral calculus, partial derivatives, maxima and minima, multiple integrals, Fourier series, vector identities, line, surface and volume integrals, Stokes, Gauss and Green's theorems.

Differential Equations : First order equation (linear and nonlinear), higher order linear differential equations with constant coefficients, method of variation of parameters, Cauchy's and Euler's equations, initial and boundary value problems, solution of partial differential equations: variable separable method.

Analysis of complex variables: : Analytic functions, Cauchy's integral theorem and integral formula, Taylor's and Laurent's series, residue theorem, solution of integrals.

Complex Variables : Analytic functions, Cauchy's integral theorem and integral formula, Taylor's and Laurent' series, Residue theorem, solution integrals.

Probability and Statistics : Sampling theorems, conditional probability, mean, median, mode and standard deviation, random variables, discrete and continuous distributions: normal, Poisson and binomial distributions.

Numerical Methods : Matrix inversion, solutions of non-linear algebraic equations, iterative methods for solving differential equations, numerical integration, regression and correlation analysis.

GATE: Computer Science & IT Engineering

Linear Algebra: Matrices, determinants, system of linear equations, eigenvalues and eigenvectors, LU decomposition.

Calculus: Limits, continuity and differentiability. Maxima and minima. Mean value theorem. Integration.

Probability: Random variables. Uniform, normal, exponential, poisson and binomial distributions. Mean, median, mode and standard deviation. Conditional probability and Bayes theorem.

Contents

1 Linear Algebra

1.1 Introduction

Linear Algebra is a branch of mathematics concerned with the study of vectors, with families of vectors called vector spaces or linear spaces and with functions that input one vector and output another, according to certain rules. These functions are called linear maps or linear transformations and are often represented by matrices. Matrices are rectangular arrays of numbers or symbols and matrix algebra or linear algebra provides the rules defining the operations that can be formed on such an object.

Linear Algebra and matrix theory occupy an important place in modern mathematics and has applications in almost all branches of engineering and physical sciences. An elementary application of linear algebra is to the solution of a system of linear equations in several unknowns, which often result when linear mathematical models are constructed to represent physical problems. Nonlinear models can often be approximated by linear ones. Other applications can be found in computer graphics and in numerical methods.

In this chapter, we shall discuss matrix algebra and its use in solving linear system of algebraic equations $AX = B$ and in solving the Eigen value problem $AX = \lambda X$.

1.2 Algebra of Matrices

1.2.1 Definition of Matrix

A system of $m \times n$ numbers arranged in the form of a rectangular array having m rows and n columns is called an matrix of order $m \times n$.

If $A = [a_{ij}]_{m\times n}$ be any matrix of order $m \times n$ then it is written in the form:

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12}\ldots\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22}\ldots\ldots\ldots\ldots a_{2n} \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2}\ldots\ldots\ldots\ldots a_{mn} \end{bmatrix}$$

Horizontal lines are called rows and vertical lines are called columns.

1.2.2 Special Types of Matrices

1. **Square Matrix:** An $m\times n$ matrix for which $m = n$ (The number of rows is equal to number of columns) is called square matrix. It is also called an n-rowed square matrix. i.e. $A = [a_{ij}]_{n\times n}$. The elements $a_{ij} \mid i = j$, i.e. a_{11}, a_{22}.... are called DIAGONAL ELEMENTS and the line along which they lie is called PRINCIPLE DIAGONAL of matrix. Elements other than a_{11}, a_{22}, etc are called off-diagonal elements i.e. $a_{ij} \mid i \neq j$.

   Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 9 & 8 & 3 \end{bmatrix}_{3\times 3}$ is a square Matrix

NOTE

A square sub-matrix of a square matrix A is called a **"principle sub-matrix"** if its diagonal elements are also the diagonal elements of the matrix A. So $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ is a principle sub matrix of the matrix A given above, but $\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$ is not.

2. **Diagonal Matrix:** A square matrix in which all off-diagonal elements are zero is called a diagonal matrix. The diagonal elements may or may not be zero. $\begin{cases} a_{ij} = 0 & \text{if } i \neq j \\ a_{ij} & \text{if } i = j \end{cases}$

Example: $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ is a diagonal matrix

The above matrix can also be written as A = diag [3, 5, 9]

Properties of Diagonal Matrix:

diag $[x, y, z]$ + diag $[p, q, r]$ = diag $[x + p, y + q, z + r]$

diag $[x, y, z] \times$ diag $[p, q, r]$ = diag $[xp, yq, zr]$

(diag $[x, y, z])^{-1}$ = diag $[1/x, 1/y, 1/z]$

(diag $[x, y, z])^T$ = diag $[x, y, z]$

(diag $[x, y, z])^n$ = diag $[x^n, y^n, z^n]$

Eigen values of diag $[x, y, z] = x, y$ and z.

Determinant of diag $[x, y, z] = |\text{diag } [x, y, z]| = xyz$

3. **Scalar Matrix:** A scalar matrix is a diagonal matrix with all diagonal elements being equal. $\begin{cases} a_{ij} = 0 & \text{if } i \neq j \\ a_{ij} = k & \text{if } i = j \end{cases}$

Example: $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a scalar matrix.

4. **Unit Matrix or Identity Matrix:** A square matrix each of whose diagonal elements is 1 and each of whose non-diagonal elements are zero is called unit matrix or an identity matrix which is denoted by I. Identity matrix is always square.

Thus a square matrix $A = [a_{ij}]$ is a unit matrix if $a_{ij} = 1$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$. $\begin{cases} a_{ij} = 0 & \text{if } i \neq j \\ a_{ij} = 1 & \text{if } i = j \end{cases}$

Example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is unit matrix, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Properties of Identity Matrix:

(a) I is Identity element for multiplication, so it is called multiplicative identity.

(b) $AI = IA = A$

(c) $I^n = I$

(d) $I^{-1} = I$

(e) $|I| = 1$

5. Null Matrix: The $m \times n$ matrix whose elements are all zero is called null matrix.

Null matrix is denoted by O. Null matrix need not be square. $a_{ij} = 0 \;\forall\; i, j$

Example: $O_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $O_{2\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Properties of Null Matrix:

(a) $A + O = O + A = A$

So, O is additive identity.

(b) $A + (-A) = O$

6. Upper Triangular Matrix: An upper triangular matrix is a square matrix whose lower off-diagonal elements are zero, i.e. $a_{ij} = 0$ whenever $i > j$

It is denoted by U.

The diagonal and upper off diagonal elements may or may not be zero. $\begin{cases} a_{ij} = 0 & \text{if } i > j \\ a_{ij} & \text{if } i < j \end{cases}$

Example: $U = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 5 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

7. Lower Triangular Matrix: A lower triangular matrix is a square matrix whose upper off-diagonal triangular elements are zero, i.e. $a_{ij} = 0$ whenever $i < j$. The diagonal and lower off-diagonal elements may or may not be zero. $\begin{cases} a_{ij} = 0 & \text{if } i < j \\ a_{ij} & \text{if } i > j \end{cases}$

It is denoted by L,

Example: $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 5 & 0 \\ 2 & 3 & 6 \end{bmatrix}$

8. Idempotent Matrix: A matrix A is called Idempotent if $A^2 = A$.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ are examples of Idempotent matrices.

9. Involuntary Matrix: A matrix A is called Involutory if $A^2 = I$.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is involuntary. Also $\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ is involuntary since $A^2 = I$.

10. **Nilpotent Matrix:** A matrix A is said to be nilpotent of class x or index x if $A^x = O$ and $A^{x-1} \neq O$ i.e. x is the smallest index which makes $A^x = O$.

Example: The matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent class 3, since $A \neq 0$ and $A^2 \neq 0$, but $A^3 = 0$.

11. **Singular matrix:** If the determinant of a matrix is zero, then matrix is called as singular matrix.

$|A| = 0$ e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

**If determinant is not zero, then matrix is known as non-singular matrix.*

If matrix is singular then its inverse doesn't exist 3.

1.2.3 Equality of Two Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if,

1. They are of same size.
2. The elements in the corresponding places of two matrices are the same i.e., $a_{ij} = b_{ij}$ for each pair of subscripts i and j.

Example: Let $\begin{bmatrix} x-y & p+q \\ p-q & x+y \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 10 \end{bmatrix}$

Then $x - y = 2$, $p + q = 5$, $p - q = 1$ and $x + y = 10$

$\Rightarrow$ $x = 6$, $y = 4$, $p = 3$ and $q = 2$.

1.2.4 Addition of Matrices

Two matrices A and B are compatible for addition only if they both have exactly the same size say $m \times n$. Then their sum is defined to be the matrix of the type $m \times n$ obtained by adding corresponding elements of A and B. Thus if, $A = [a_{ij}]_{m \times n}$ & $B = [b_{ij}]_{m \times n}$ then $A + B = [a_{ij} + b_{ij}]_{m \times n}$.

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix}$;

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 10 & 13 \end{bmatrix}$$

Properties of Matrix Addition:

1. Matrix addition is commutative $A + B = B + A$.
2. Matrix addition is associative $(A + B) + C = A + (B + C)$
3. Existence of additive identity: If O be $m \times n$ matrix each of whose elements are zero. Then, $A + O = A = O + A$ for every $m \times n$ matrix A.
4. Existence of additive inverse: Let $A = [a_{ij}]_{m \times n}$.

   Then the negative of matrix A is defined as matrix $[-a_{ij}]_{m \times n}$ and is denoted by $-A$.

   $\Rightarrow$ Matrix $-A$ is additive inverse of A. Because $(-A) + A = O = A + (-A)$. Here O is null matrix of order $m \times n$.
5. Cancellation laws holds good in case of addition of matrices, which is $X = -A$.

   $A + X = B + X \Rightarrow A = B$

   $X + A = X + B \Rightarrow A = B$
6. The equation $A + X = 0$ has a unique solution in the set of all $m \times n$ matrices.

1.2.5 Substraction of Two Matrices

If A and B are two $m \times n$ matrices, then we define, $A - B = A + (-B)$.

Thus the difference $A - B$ is obtained by subtracting from each element of A corresponding elements of B.

NOTE: Subtraction of matrices is neither commutative nor associative.

1.2.6 Multiplication of a Matrix by a Scalar

Let A be any $m \times n$ matrix and k be any real number called scalar. The $m \times n$ matrix obtained by multiplying every element of the matrix A by k is called scalar multiple of A by k and is denoted by kA.

$\Rightarrow$ If $A = [a_{ij}]_{m \times n}$ then $Ak = kA = [kA]_{m \times n}$.

If $A = \begin{bmatrix} 5 & 2 & 1 \\ 6 & -5 & 2 \\ 1 & 3 & 6 \end{bmatrix}$ then, $3A = \begin{bmatrix} 15 & 6 & 3 \\ 18 & -15 & 6 \\ 3 & 9 & 18 \end{bmatrix}$

Properties of Multiplication of a Matrix by a Scalar:

1. Scalar multiplication of matrices distributes over the addition of matrices i.e., $k(A + B) = kA + kB$.
2. If p and q are two scalars and A is any $m \times n$ matrix then, $(p + q)A = pA + qA$.
3. If p and q are two scalers and $A = [a_{ij}]_{m \times n}$ then, $p(qA) = (pq)A$.
4. If $A = [a_{ij}]_{m \times n}$ be a matrix and k be any scalar then, $(-k)A = -(kA) = k(-A)$.

1.2.7 Multiplication of Two Matrices

Let $A = [a_{ij}]_{m \times n}$; $B = [b_{jk}]_{n \times p}$ be two matrices such that the number of columns in A is equal to the number of rows in B.

Then the matrix $C = [c_{ik}]_{m \times p}$ such that $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$ is called the product of matrices A and B in that order and we write $C = AB$.

Properties of Matrix Multiplication:

1. Multiplication of matrices is not commutative. In fact, if the product of AB exists, then it is not necessary that the product of BA will also exist. For example, $A_{3 \times 2} \times B_{2 \times 4} = C_{3 \times 4}$ but $B_{2 \times 4} \times A_{3 \times 2}$ does not exist since these are not compatible for multiplication.
2. Matrix multiplication is associative, if conformability is assured. i.e., $A(BC) = (AB)C$ where A, B, C are $m \times n$, $n \times p$, $p \times q$ matrices respectively.
3. Multiplication of matrices is distributive with respect to addition of matrices. i.e., $A(B + C) = AB + AC$.
4. The equation $AB = O$ does not necessarily imply that at least one of matrices A and B must be a zero matrix. For example, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
5. In the case of matrix multiplication if $AB = O$ then it is not necessarily imply that $BA = O$. In fact, BA may not even exist.
6. Both left and right cancellation laws hold for matrix multiplication as shown below:

   $AB = AC \Rightarrow B = C$ (if A is non-singular matrix) and

   $BA = CA \Rightarrow B = C$ (if A is non-singular matrix).

1.2.8 Trace of a Matrix

Let A be a square matrix of order n. The sum of the elements lying along principal diagonal is called the trace of A denoted by $Tr(A)$.

Thus if $A = [a_{ij}]_{n \times n}$ then, $Tr(A) = \sum_{i=1}^{n} a_{ij} = a_{11} + a_{22} + \dots a_{nn}$.

Let $$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 6 & 5 \end{bmatrix}$$

Then, Trace $(A) = Tr(A) = 1+(-3) + 5 = 3$

Properties of Trace of a Matrix:

Let A and B be two square matrices of order n and λ be a scalar. Then,

1. $Tr(\lambda A) = \lambda \, Tr\, A$
2. $Tr(A + B) = Tr\, A + Tr\, B$
3. $Tr(AB) = Tr(BA)$ [If both AB and BA are defined]

1.2.9 Transpose of a Matrix

Let $A = [a_{ij}]_{m\times n}$. Then the $n \times m$ matrix obtained from A by changing its rows into columns and its columns into rows is called the transpose of A and is denoted by A' or A^T.

Let $$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 6 & 5 \end{bmatrix} \text{ then, } A^T = A' = \begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{bmatrix}$$

If $$B = [1\ 2\ 3]$$

Then $$B' = [1 \quad 2 \quad 3]' = [1 \quad 2 \quad 3]^t = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Properties of Transpose of a Matrix:

If A^T and B^T be transposes of A and B respectively then,

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(kA)^T = kA^T$, k being any complex number
4. $(AB)^T = B^T A^T$
5. $(ABC)^T = C^T B^T A^T$

1.2.10 Conjugate of a Matrix

The matrix obtained from given matrix A on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\overline{A}$.

Example: If $A = \begin{bmatrix} 2+3i & 4-7i & 8 \\ -i & 6 & 9+i \end{bmatrix}$

$$\overline{A} = \begin{bmatrix} 2-3i & 4+7i & 8 \\ +i & 6 & 9-i \end{bmatrix}$$

Properties of Conjugate of a Matrix:

If $\overline{A}$ and $\overline{B}$ be the conjugates of A and B respectively. Then,

1. $\overline{(\overline{A})} = A$
2. $\overline{(A+B)} = \overline{A}+\overline{B}$
3. $\overline{(kA)} = \overline{k}\,\overline{A}$, k being any complex number
4. $\overline{(AB)} = \overline{A}\,\overline{B}$, A and B being conformable to multiplication
5. $\overline{A} = A$ if A is real matrix

   $\overline{A} = -A$ if A is purely imaginary matrix

1.2.11 Transposed Conjugate of Matrix

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^θ or A^* or $(\overline{A})^T$. It is also called conjugate transpose of A.

Example: If $A = \begin{bmatrix} 2+i & 3-i \\ 4 & 1-i \end{bmatrix}$

To find A^θ, we first find $\overline{A} = \begin{bmatrix} 2-i & 3+i \\ 4 & 1+i \end{bmatrix}$

Then $A^\theta = (\overline{A})^T = \begin{bmatrix} 2-i & 4 \\ 3+i & 1+i \end{bmatrix}$

Some properties: If A^θ & B^θ be the transposed conjugates of A and B respectively then,

1. $(A^\theta)^\theta = A$
2. $(A + B)^\theta = A^\theta + B^\theta$
3. $(kA)^\theta = \overline{k}A^\theta$, $k \to$ complex number
4. $(AB)^\theta = B^\theta A^\theta$

1.2.12 Classification of Real Matrices

Real matrices can be classified into the following three types based on the relationship between A^T and A.

1. Multip

1. Symmetric Matrices ($A^T = A$)
2. Skew Symmetric Matrices ($A^T = -A$)
3. Orthogonal Matrices ($A^T = A^{-1}$ or $AA^T = I$)

1. **Symmetric Matrix:** A square matrix $A = [a_{ij}]$ is said to be symmetric if its $(i, j)^{th}$ elements is same as its $(j, i)^{th}$ element i.e., $a_{ij} = a_{ji}$ for all i & j.

   In a symmetric matrix, $A^T = A$

   Example: $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix, since $A^T = A$.

   Note: For any matrix A,

   (a) AA^T is always a symmetric matrix.

(b) $\dfrac{A+A^T}{2}$ is always symmetric matrix.

Note: If A and B an symmetric, then

(a) $A + B$ and $A - B$ are also symmetric.

(b) AB, BA may or may not be symmetric.

2. Skew Symmetric Matrix: A square matrix $A = [a_{ij}]$ is said to be skew symmetric if $(i, j)^{th}$ elements of A is the negative of the $(j, i)^{th}$ elements of A if $a_{ij} = -a_{ji}\ \forall\ i, j$.

In a skew symmetric matrix $A^T = -A$.

A skew symmetric matrix must have all 0's in the diagonal.

Example: $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Note: For any matrix A, the matrix $\dfrac{A-A^T}{2}$ is always skew symmetric.

3. Orthogonal Matrix: A square matrix A is said be orthogonal if:

$A^T = A^{-1} \Rightarrow AA^T = AA^{-1} = I$. Thus A will be an orthogonal matrix if, $AA^T = I = A^TA$.

Example: The identity matrix is orthogonal since $I^T = I^{-1} = I$.

Note: Since for an orthogonal matrix A,

$$AA^T = I$$
$$\Rightarrow \quad |AA^T| = |I| = 1$$
$$\Rightarrow \quad |A|\ |A^T| = 1$$
$$\Rightarrow \quad (|A|)^2 = 1$$
$$\Rightarrow \quad |A| = \pm 1$$

So the determinant of an orthogonal matrix always has a modulus of 1.

1.2.13 Classification of Complex Matrices

Complex matrices can be classified into the following three types based on relationship between A^θ and A.

1. Hermitian Matrix ($A^\theta = A$)
2. Skew-Hermitian Matrix ($A^\theta = -A$)
3. Unitary Matrix ($A^\theta = A^{-1}$ or $AA^\theta = I$)

1. Hermitian Matrix: A necessary and sufficient condition for a matrix A to be Hermitian is that $A^\theta = A$.

Example: $A = \begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$ is a Hermitian matrix.

2. Skew-Hermitian Matrix: A necessary and sufficient condition for a matrix to be skew-Hermitian if $A^\theta = -A$.

Example: $A = \begin{bmatrix} 0 & -2-i \\ 2-i & 0 \end{bmatrix}$ is skew-Hermitian.

3. Unitary Matrix: A square matrix A is said to be unitary if:

$$A^\theta = A^{-1}$$

Multiplying both sides by A, we get an alternate definition of unitary matrix as given below:

A square matrix A is said to be unitary if:

$$AA^{\theta} = I = A^{\theta} A$$

Example: $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ is an example of a unitary matrix.

1.3 Determinants

1.3.1 Definition

Let a_{11}, a_{12}, a_{21}, a_{22} be any four numbers. The symbol $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ represents the number $a_{11}a_{22} - a_{12}a_{21}$ and is called determinants of order 2. The number a_{11}, a_{12}, a_{21}, a_{22} are called elements of the determinant and the number $a_{11}a_{22} - a_{21}a_{12}$ is called the value of determinant.

1.3.2 Minors, Cofactors and Adjoint

Consider the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Leaving the row and column passing through the elements a_{ij}, then the second order determinant thus obtained is called the minor of element a_{ij} and we will be denoted by M_{ij}.

Example: The Minor of element $a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$

Similarly Minor of element $a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = M_{32}$

1.3.3 Cofactors

The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of element a_{ij}. We shall denote the cofactor of an element by corresponding capital letter.

Example: Cofactor of $a_{ij} = A_{ij} = (-1)^{i+j} M_{ij}$.

Cofactor of element $a_{21} = A_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

by cofactor of element $a_{32} = A_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

We define for any matrix, the sum of the products of the elements of any row or column with corresponding cofactors is equal to the determinant of the matrix.

When A is diagonalisable $A = M^{-1}DM$, where the matrix D is a diagonal matrix constructed using the eigen values of A as its diagonal elements. Also the corresponding matrix M can be obtained by constructing a $n \times n$ matrix whose columns are the eigen vectors of A.

Practical application of Diagonalisation:

One of the uses of diagonalisation is for computing higher powers of a matrix efficiently.

If $A = M^{-1}DM$ then $A^n = M^{-1}D^n M$

The above property makes it easy to compute higher powers of a matrix A, since computing D^n is much more easy compared with computing A^n.

■■■■

Previous GATE and ESE Questions

Q.1 Given Matrix $[A] = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$, the rank of the matrix is

(a) 4 (b) 3
(c) 2 (d) 1

[CE, GATE-2003, 1 mark]

Q.2 Consider the system of simultaneous equations

$$x + 2y + z = 6$$
$$2x + y + 2z = 6$$
$$x + y + z = 5$$

This system has

(a) unique solution
(b) infinite number of solutions
(c) no solution
(d) exactly two solutions

[ME, GATE-2003, 2 marks]

Q.3 Consider the following system of linear equations

$$\begin{bmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 5 \\ 7 \end{bmatrix}$$

Notice that the second and the third columns of the coefficient matrix are linearly dependent. For how many values of α, does this system of equations have infinitely many solutions?

(a) 0 (b) 1
(c) 2 (d) infinitely many

[CS, GATE-2003, 2 marks]

Q.4 For the matrix $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ the eigen values are

(a) 3 and – 3 (b) –3 and – 5
(c) 3 and 5 (d) 5 and 0

[ME, GATE-2003, 1 mark]

Q.5 For which value of x will the matrix given below become singular?

$$\begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$$

(a) 4 (b) 6
(c) 8 (d) 12

[ME, GATE-2004, 2 marks]

Q.6 Let A, B, C, D be $n \times n$ matrices, each with non-zero determinant, If $ABCD = I$, then B^{-1} is

(a) $D^{-1} C^{-1} A^{-1}$
(b) CDA
(c) ADC
(d) does not necessarily exist

[CS, GATE-2004, 1 mark]

Q.7 How many solutions does the following system of linear equations have?

$-x + 5y = -1$; $x - y = 2$; $x + 3y = 3$

(a) infinitely many (b) two distinct solutions
(c) unique (d) none

[CS, GATE-2004, 2 marks]

Q.8 The eigen values of the matrix $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$

(a) are 1 and 4 (b) are –1 and 2
(c) are 0 and 5 (d) cannot be determined

[CE, GATE-2004, 2 marks]

Q.9 The sum of the eigen values of the matrix given below is $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

(a) 5 (b) 7
(c) 9 (d) 18

[ME, GATE-2004, 1 mark]

Q.10 Consider the matrices $X_{(4 \times 3)}$, $Y_{(4 \times 3)}$ and $P_{(2 \times 3)}$. The order of $[P(X^TY)^{-1} P^T]^T$ will be

(a) (2×2) (b) (3×3)
(c) (4×3) (d) (3×4)

[CE, GATE-2005, 1 mark]

Q.11 Given an orthogonal matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \ \left[AA^T\right]^{-1} \text{ is}$$

(a) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$

[EC, GATE-2005, 2 marks]

Q.12 If R = $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$, then top row of R^{-1} is

(a) [5 6 4] (b) [5 –3 1]
(c) [2 0 –1] (d) [2 –1 1/2]

[EE, GATE-2005, 2 marks]

Q.13 Let, $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} \frac{1}{2} & a \\ 0 & b \end{bmatrix}$.

Then $(a + b) =$

(a) $\frac{7}{20}$ (b) $\frac{3}{20}$
(c) $\frac{19}{60}$ (d) $\frac{11}{20}$

[EC, GATE-2005, 2 marks]

Q.14 Consider a non-homogeneous system of linear equations representing mathematically an over-determined system. Such a system will be

(a) consistent having a unique solution
(b) consistent having many solutions
(c) inconsistent having a unique solution
(d) inconsistent having no solution

[CE, GATE-2005, 1 mark]

Q.15 A is a 3 × 4 real matrix and $A x = b$ is an inconsistent system of equations. The highest possible rank of A is

(a) 1 (b) 2
(c) 3 (d) 4

[ME, GATE-2005, 1 mark]

Q.16 In the matrix equation $Px = q$, which of the following is a necessary condition for the existence of at least one solution for the unknown vector x

(a) Augmented matrix $[Pq]$ must have the same rank as matrix P
(b) Vector q must have only non-zero elements
(c) Matrix P must be singular
(d) Matrix P must be square

[EE, GATE-2005, 1 mark]

Q.17 Consider the following system of equations in three real variables x_1, x_2 and x_3

$$2x_1 - x_2 + 3x_3 = 1$$
$$3x_1 - 2x_2 + 5x_3 = 2$$
$$-x_1 - 4x_2 + x_3 = 3$$

This system of equations has

(a) no solution

(b) a unique solution

(c) more than one but a finite number of solutions

(d) an infinite number of solutions

[CS, GATE-2005, 2 marks]

Q.18 Which one of the following is an eigen vector of the matrix $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$

[ME, GATE-2005, 2 marks]

Q.19 For the matrix $A = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, one of the eigen values is equal to –2. Which of the following is an eigen vector?

(a) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

[EE, GATE-2005, 2 marks]

Q.20 Given the matrix $\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$, the eigen vector is

(a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

[EC, GATE-2005, 2 marks]

Q.21 What are the eigen values of the following 2×2 matrix?

$$\begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

(a) –1 and 1 (b) 1 and 6

(c) 2 and 5 (d) 4 and –1

[CS, GATE-2005, 2 marks]

Q.22 Consider the system of equations $A_{(n \times n)} x_{(n \times t)} = \lambda_{(n \times l)}$ where, λ is a scalar. Let (λ_i, x_i) be an eigen-pair of an eigen value and its corresponding eigen vector for real matrix A. Let I be a $(n \times n)$ unit matrix. Which one of the following statement is NOT correct?

(a) For a homogeneous $n \times n$ system of linear equations, $(A - \lambda I)x = 0$ having a nontrivial solution, the rank of $(A - \lambda I)$ is less than n

(b) For matrix A^m, m being a positive integer, (λ_i^m, x_i^m) will be the eigen-pair for all i

(c) If $A^T = A^{-1}$, then $|\lambda_i| = 1$ for all i

(d) If $A^T = A$, then λ_i is real for all i

[CE, GATE-2005, 2 marks]

Q.23 Multiplication of matrices E and F is G. Matrices E and G are

$$E \equiv \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } G \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

What is the matrix F?

(a) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} \cos\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

[ME, GATE-2006, 2 marks]

Explanations Linear Algebra

1. (c)

Consider first 3 × 3 minors, since maximum possible rank is 3

$$\begin{vmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 4 & 7 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 1 & 3 \\ 6 & 4 & 7 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

and $$\begin{vmatrix} 4 & 2 & 3 \\ 6 & 3 & 7 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

Since all 3 × 3 minors are zero, now try 2 × 2 minors.

$$\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5 \neq 0$$

So, rank = 2

2. (c)

Given equation are

$$x + 2y + z = 6$$
$$2x + y + 2z = 6$$
$$x + y + z = 5$$

Given system can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 5 \end{bmatrix}$$

Augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 2 & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \end{array}\right]$

By gauss elimination

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 2 & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & -3 & 0 & -6 \\ 0 & -1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 - \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$r(A) = 2$$
$$r(A \mid B) = 3$$

Since the rank of coefficient matrix is 2 and rank of argument matrix is 3, which is not equal. Hence system has no solution i.e. system is inconsistent.

3. (b)

The augmented matrix for the given system is

$$\left[\begin{array}{ccc|c} 2 & 1 & -4 & \alpha \\ 4 & 3 & -12 & 5 \\ 1 & 2 & -8 & 7 \end{array}\right].$$

Performing Gauss-Elimination on the above matrix

$$\left[\begin{array}{ccc|c} 2 & 1 & -4 & \alpha \\ 4 & 3 & -12 & 5 \\ 1 & 2 & -8 & 7 \end{array}\right] \xrightarrow[R_3 - 1/2\,R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 2 & 1 & -4 & \alpha \\ 0 & 1 & -4 & 5-2\alpha \\ 0 & 3/2 & -6 & 7-\alpha/2 \end{array}\right]$$

$$\xrightarrow{R_3 - 3/2\,R_2} \left[\begin{array}{ccc|c} 2 & 1 & -4 & \alpha \\ 0 & 1 & -4 & 5-2\alpha \\ 0 & 0 & 0 & \frac{5\alpha - 1}{2} \end{array}\right]$$

Now for infinite solution it is necessary that at least one row must be completely zero.

$$\therefore \quad \frac{5\alpha - 1}{2} = 0$$

α = 1/5 is the solution

$\therefore$ There is only one value of α for which infinite solution exists.

4. (c)

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

Now, $A - \lambda I = 0$

Where λ = eigen value

$$\therefore \quad \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix} = 0$$

$$(4-\lambda)^2 - 1 = 0$$

or, $(4-\lambda)^2 - (1)^2 = 0$

or, $(4 - \lambda + 1)(4 - \lambda - 1) = 0$

or, $(5-\lambda)(3-\lambda) = 0$

$\therefore \quad \lambda = 3, \lambda = 5$

5. (a)

For singularity of matrix $= \begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix} = 0$

$\Rightarrow 8(0 - 12) - x(0 - 2 \times 12) = 0$

$\therefore \quad x = 4$

6. (b)

A, B, C, D is $n \times n$ matrix.

Given $ABCD = I$

$$\Rightarrow \quad ABCDD^{-1}C^{-1} = D^{-1}C^{-1}$$
$$\Rightarrow \quad AB = D^{-1}C^{-1}$$
$$\Rightarrow \quad A^{-1}AB = A^{-1}D^{-1}C^{-1}$$
$$\Rightarrow \quad B = A^{-1}D^{-1}C^{-1}$$
$$B^{-1} = (A^{-1}D^{-1}C^{-1})^{-1}$$
$$= (C^{-1})^{-1} \cdot (D^{-1})^{-1} \cdot (A^{-1})^{-1}$$
$$= CDA$$

7. (c)

$$-x + 5y = -1$$
$$x - y = 2$$
$$x + 3y = 3$$

The avg mented matrix is $\left[\begin{array}{cc|c} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{array}\right]$.

Using gauss-elimination on above matrix we get,

$$\left[\begin{array}{cc|c} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{array}\right] \xrightarrow[R_3 + R_1]{R_2 + R_1} \left[\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Rank $[A \mid B] = 2$ (number of non zero rows in $[A \mid B]$)

Rank $[A] = 2$ (number of non zero rows in [A])

Rank $[A \mid B]$ = Rank $[A]$

= 2 = number of variables

$\therefore$ Unique solution exists. Correct choice is (c).

8. (c)

Characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda) \times (1-\lambda) - [(-2) \times (-2)] = 0$$
$$\lambda^2 - 5\lambda = 0$$
$$\Rightarrow \quad \lambda(\lambda - 5) = 0$$

Hence, $\lambda = 0, 5$ are the eigen values.

9. (b)

Sum of eigen values of given matrix = sum of diagonal element of given matrix = $1 + 5 + 1 = 7$.

10. (a)

With the given order we can say that order of matrices are as follows:

$$X^T \to 3 \times 4$$
$$Y \to 4 \times 3$$
$$X^T Y \to 3 \times 3$$
$$(X^T Y)^{-1} \to 3 \times 3$$
$$P \to 2 \times 3$$
$$P^T \to 3 \times 2$$
$$P(X^T Y)^{-1} P^T \to (2\times3)(3 \times 3)(3 \times 2) \to 2 \times 2$$
$$\therefore \quad (P(X^T Y)^{-1} P^T)^T \to 2 \times 2$$

11. (c)

For orthogonal matrix

$AA^T = I$ i.e. Identity matrix.

$$\therefore \quad (AA^T)^{-1} = I^{-1} = I$$

12. (b)

$$R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$$

$$R^{-1} = \frac{\text{adj}(R)}{|R|} = \frac{[\text{cofactor } (R)]^T}{|R|}$$

$$|R| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{vmatrix}$$
$$= 1(2+3) - 0(4+2) - 1(6-2)$$
$$= 5 - 4 = 1$$

Since we need only the top row of R^{-1}, we need to find only first column of cof (R) which after transpose will become first row of adj (R).

$$\text{cof. } (1, 1) = +\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 2 + 3 = 5$$

$$\text{cof. } (2, 1) = -\begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = -3$$

$$\text{cof. } (3, 1) = +\begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} = +1$$

$$\therefore \quad \text{cof. } (A) = \begin{bmatrix} 5 & - & - \\ -3 & - & - \\ 1 & - & - \end{bmatrix}$$

$$\text{Adj } (A) = [\text{cof. } (A)]^T = \begin{bmatrix} 5 & -3 & 1 \\ - & - & - \\ - & - & - \end{bmatrix}$$

Dividing by $|R| = 1$ gives

$$R^{-1} = \begin{bmatrix} 5 & -3 & 1 \\ - & - & - \\ - & - & - \end{bmatrix}$$

$\therefore$ Top row of $R^{-1} = [5 \;\; -3 \;\; 1]$

13. (a)

$$[AA^{-1}] = I$$

$$\Rightarrow \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1/2 & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2a-0.1b \\ 0 & 3b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 2a - 0.1b = 0 \Rightarrow a = \frac{0.1b}{2} \quad \text{...(i)}$$

$$3b = 1 \Rightarrow b = \frac{1}{3}$$

Now substitute b in equation (i), we get

$$a = \frac{1}{60}$$

So,
$$a + b = \frac{1}{60} + \frac{1}{3}$$

$$= \frac{1+20}{60} = \frac{21}{60} = \frac{7}{20}$$

14. (a), (b) and (d) all possible.
In an over determined system having more equations than variables, all three possibilities still exist (a) consistent unique (b) consistent infinite and (d) in consistent with no situation.

15. (b)

$$r(A_{m \times n}) \le \min(m, n)$$

So, Highest possible rank = Least value of 3 and 4.
i.e. highest possible rank (based on size of A) = 3
However if the rank of A = 3 then rank of $[A \mid B]$ also would be 3, which means the system would become consistent. But it is given that the system is inconsistent. So the maximum rank of A could only be 2.

16. (a)
Rank $[Pq]$ = Rank $[P]$ is necessary for existence of at least one solution to $Px = q$.

17. (b)
The augmented matrix for the given system is

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 3 & -2 & 5 & 2 \\ -1 & -4 & 1 & 3 \end{array}\right].$$

Using gauss-elimination method on above matrix we get,

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 3 & -2 & 5 & 2 \\ -1 & -4 & 1 & 3 \end{array}\right] \xrightarrow[R_3 + \frac{1}{2}R_1]{R_2 - \frac{3}{2}R_1} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & -1/2 & 1/2 & 1/2 \\ 0 & -9/2 & 5/2 & 7/2 \end{array}\right]$$

$$\xrightarrow{R_3 - 9R_2} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & -2 & -1 \end{array}\right]$$

Rank $([A \mid B]) = 3$
Rank $([A]) = 3$

Since Rank $([A \mid B])$ = Rank $([A])$ = number of variables. The system has unique solution.

18. (a)
First solve for eigen values by solving characteristic equation $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 0 & 0 & 0 \\ 0 & 5-\lambda & 5 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$= (5-\lambda)(5-\lambda)[(2-\lambda)(1-\lambda) - 3]$$
$$= 0$$
$$= (5-\lambda)(5-\lambda)(\lambda^2 - 3\lambda - 1) = 0$$

$$\lambda = 5, 5, \frac{3 \pm \sqrt{13}}{2}$$

put $\lambda = 5$ in $[A - \lambda I]X = 0$

$$\begin{bmatrix} 5-5 & 0 & 0 & 0 \\ 0 & 5-5 & 5 & 0 \\ 0 & 0 & 2-5 & 1 \\ 0 & 0 & 3 & 1-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 5x_3 = 0$; $-3x_3 + x_4 = 0$; $3x_3 - 4x_4 = 0$
Solving which we get $x_3 = 0$, $x_4 = 0$, x_1 and x_2 may be anything.
The eigen vector corresponding to $\lambda = 5$, may be written as

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ 0 \\ 0 \end{bmatrix}$$

where k_1, k_2 may be any real number. Since choice (a) is the only matrix in this form with both x_3 and $x_4 = 0$, so it is the correct answer.
Since, we already got a correct eigen vector, there is no need to derive the eigen vector corresponding to $\lambda = \frac{3 \pm \sqrt{13}}{2}$.

19. (d)

Since matrix is triangular, the eigen values are the diagonal elements themselves namely λ = 3, –2 and 1. Corresponding to eigen value, λ = –2 let us find the eigen vector

$$[A - \lambda I]\,x = 0$$

$$\begin{bmatrix} 3-\lambda & -2 & 2 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting λ = –2 in above equation we get,

$$\begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which gives the equations,

$$5x_1 - 2x_2 + 2\,x_3 = 0 \quad \text{... (i)}$$
$$x_3 = 0 \quad \text{... (ii)}$$
$$3\,x_3 = 0 \quad \text{... (iii)}$$

Since eq. (ii) and (iii) are same we have

$$5x_1 - 2x_2 + 2x_3 = 0 \quad \text{... (i)}$$
$$x_3 = 0 \quad \text{... (ii)}$$

Putting x_2 = k, in eq. (i) we get

$5x_1 - 2\,k + 2 \times 0 = 0$

$\Rightarrow \quad x_1 = 2/5\,k$

$\therefore$ Eigen vectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/5\,k \\ k \\ 0 \end{bmatrix}$$

i.e. $x_1 : x_2 : x_3 = 2/5\,k : k : 0 = 2/5 : 1 : 0 = 2 : 5 : 0$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$ is an eigen vector of matrix *A*.

20. (c)

First, find the eigen values of $A = \begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -4-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (-4 - \lambda)(3 - \lambda) - 8 = 0$

$\Rightarrow \quad \lambda^2 + \lambda - 20 = 0$

$\Rightarrow \quad (\lambda + 5)(\lambda - 4) = 0$

$\Rightarrow \quad \lambda_1 = -5$ and $\lambda_2 = 4$

Corresponding to λ_1 = –5 we need to find eigen vector:

The eigen value problem is $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} -4-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} = 0$$

Putting λ = –5

we get, $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$x_1 + 2x_2 = 0 \quad \text{... (i)}$$
$$4x_1 + 8x_2 = 0 \quad \text{... (ii)}$$

Since (i) and (ii) are the same equation we take

$$x_1 + 2x_2 = 0$$
$$x_1 = -2x_2$$
$$x_1 : x_2 = -2 : 1$$

$$\Rightarrow \quad \frac{x_1}{x_2} = -2$$

Now from the answers given, we look for any vector in this ratio and we find choice (c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is in this ratio $\frac{x_1}{x_2} = \frac{2}{-1} = -2$.

So choice (c) is an eigen vector corresponding to λ = –5.

Since we already got an answer, there is no need to find the second eigen vector corresponding to λ = 4.

21. (b)

$$A = \begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

The characteristic equation of this matrix is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -4 & 5-\lambda \end{vmatrix} = 0$$

$$(2 - \lambda)(5 - \lambda) - 4 = 0$$
$$\lambda^2 - 7\lambda + 6 = 0$$
$$\lambda = 1, 6$$

$\therefore$ The eigen values of A are 1 and 6.

22. (b)

Although λ_i^m will be the corresponding eigen values of A^m, x_i^m need not be corresponding eigen vectors.