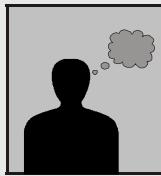


**ESE
GATE**

**MADE EASY
workbook 2023**



**Detailed Explanations of
Try Yourself Questions**

Engineering Mathematics



MADE EASY
— Publications

1

Linear Algebra



Detailed Explanation of Try Yourself Questions

T1 : Solution

(a)

With the given order we can say that order of matrices are as follows:

$$\begin{aligned} X^T &\rightarrow 3 \times 4 \\ Y &\rightarrow 4 \times 3 \\ X^T Y &\rightarrow 3 \times 3 \\ (X^T Y)^{-1} &\rightarrow 3 \times 3 \\ P &\rightarrow 2 \times 3 \\ P^T &\rightarrow 3 \times 2 \\ P(X^T Y)^{-1} P^T &\rightarrow (2 \times 3) (3 \times 3) (3 \times 2) \rightarrow 2 \times 2 \\ \therefore (P(X^T Y)^{-1} P^T)^T &\rightarrow 2 \times 2 \end{aligned}$$

T2 : Solution

(b)

The matrix can be

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

Number of ways possible for one matrix = 4^4

\Rightarrow Total number of matrix possible = 3×4^4

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

T3 : Solution

(c)

$$A \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, A \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}, A \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 6 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Let,

$$T_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, T_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, T_3 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

$$AT_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, AT_2 = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}, AT_3 = \begin{bmatrix} 0 \\ -3 \\ 6 \end{bmatrix}$$

$$A[T_1 \ T_2 \ T_3] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$[T_1 \ T_2 \ T_3] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = Q$$

$$A[T_1 \ T_2 \ T_3] = AQ = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 4 & -3 \\ 0 & -2 & 6 \end{bmatrix}$$

T4 : Solution

(c)

Method 1:

$$E = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

According to problem

$$E \times F = G$$

$$\text{or } \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence we see that product of $(E \times F)$ is unit matrix so F has to be the inverse of E .

$$\begin{aligned} F &= E^{-1} = \frac{\text{Adj}(E)}{|E|} \\ &= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Method 2:

An easier method for finding F is by multiplying E with each of the choices (a), (b), (c) and (d) and finding out which one gives the product as identity matrix G . Again the answer is (c).

T5 : Solution

(a)

$$\text{For singularity of matrix } = \begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix} = 0$$

$$\Rightarrow 8(0 - 12) - x(0 - 2 \times 12) = 0$$

$$\therefore x = 4$$

T6 : Solution

(b)

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} &= \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ \therefore \begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}^{-1} &= \frac{1}{[(3+2i)(3-2i)+i^2]} \begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix} \\ &= \frac{1}{12} \begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix} \end{aligned}$$

T7 : Solution

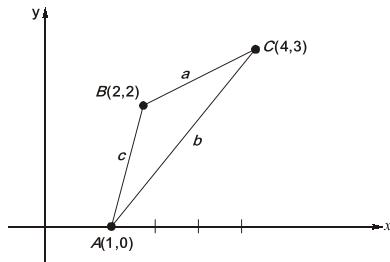
(b)

Take the determinant of given matrix $|A|$

$$\begin{aligned} &= 2[2(4-1)-1(2-1)+1(1-2)] - 1[1(4-1)-1(2-1)+1(1-2)] \\ &\quad + 1[1(2-1)-2(2-1)+1(1-1)] - 1[1(1-2)-2(1-2)+1(1-1)] \\ &= 2[6-1-1] - 1[3-1-1] + 1[1-2+0] - 1[-1+2+0] \\ &= 2(4) - 1(1) + 1(-1) - 1(1) = 8 - 1 - 1 - 1 \\ &= 5 \end{aligned}$$

T8 : Solution

(a)



Area of the triangle

$$\begin{aligned} &= \frac{1}{2} |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| \\ &= \frac{1}{2} |1(2 - 3) + 2(3 - 0) + 4(0 - 2)| = \frac{1}{2} |-1 + 6 - 8| \\ &= \frac{3}{2} \end{aligned}$$

T9 : Solution

(c)

Consider first 3×3 minors, since maximum possible rank is 3

$$\begin{vmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 4 & 7 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 1 & 3 \\ 6 & 4 & 7 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

and $\begin{vmatrix} 4 & 2 & 3 \\ 6 & 3 & 7 \\ 2 & 1 & 1 \end{vmatrix} = 0$

Since all 3×3 minors are zero, now try 2×2 minors.

$$\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5 \neq 0$$

So, rank = 2

T10 : Solution

Given :

$$A = a_{ij}$$

where,

$$a_{ij} = 5 \forall i, j$$

So,

$$A = \begin{bmatrix} 5 & 5 & 5 & \dots & \dots & \dots \\ 5 & 5 & 5 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

In a matrix if all the rows or columns are proportional or identical then its rank is always 1.

Hence, Rank = 1

T11 : SolutionGiven matrix is 4×4 skew-symmetric matrix

\therefore It is non-singular

Therefore rank of $A = 4$.**T12 : Solution**

(a)

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array} \right]$$

Using Gauss-elimination method we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array} \right] \xrightarrow{\substack{R_2 - R_1 \\ R_3 - R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & \alpha-1 & \beta-5 \end{array} \right] \xrightarrow{R_3 - \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & \alpha-2 & \beta-7 \end{array} \right]$$

Now, for infinite solution last row must be completely zero

i.e. $\alpha - 2$ and $\beta - 7 = 0$

$$\Rightarrow \alpha = 2 \text{ and } \beta = 7$$

T13 : Solution

(d)

$$x + y + z = 4 \quad \dots(1)$$

$$x - y + z = 0 \quad \dots(2)$$

$$2x + y + z = 5 \quad \dots(3)$$

Adding (1) and (2) & (2) and (3) gives

$2x + 2z = 4$ and $3x + 2z = 5$ which gives $x = 1, z = 1$ and $y = 2$

Alt: Option (b) can be eliminated since they do not satisfy 1st condition. Only (d) satisfies 3rd equation.

T14 : Solution

(b)

$$\text{Consider } (AB) = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & a & b \end{bmatrix}$$

$$R_2 - 2R_1, R_3 - 5R_1$$

$$= 2 \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & -6 & a-10 & b-10 \end{bmatrix}$$

$$R_3 - 2R_2$$

$$= \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & a-8 & b-6 \end{bmatrix}$$

$$a = 8$$

$$b = 6$$

∴ Infinite many solutions.

T15 : Solution

(c)

$$(1-\lambda)x + 2y + 3z = 0$$

$$3x + (1-\lambda)y + 2z = 0$$

$$2x + 3y + (1-\lambda)z = 0$$

∴ System of equations has non-zero solution.

∴

$$|A| = 0$$

$$\begin{vmatrix} (1-\lambda) & 2 & 3 \\ 3 & (1-\lambda) & 2 \\ 2 & 3 & (1-\lambda) \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(1-\lambda)^2 - 6] - 2[3 - 3\lambda - 4] + 3[9 - 2 + 2\lambda] = 0$$

$$\Rightarrow (1-\lambda)[\lambda^2 + 1 - 2\lambda - 6] - 2[-3\lambda - 1] + 3[7 + 2\lambda] = 0$$

$$\Rightarrow -\lambda^3 + 3\lambda^2 + 15\lambda + 18 = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - 15\lambda - 18 = 0$$

$$\Rightarrow (\lambda - 6)(\lambda^2 + 3\lambda + 3) = 0$$

∴

$$\lambda - 6 = 0 \text{ and } \lambda^2 + 3\lambda + 3 \neq 0 (D = 9 - 12 < 0)$$

∴ $\lambda = 6$ is the only possible value for which system has a non-zero solution.

T16 : Solution

(d)

For non trivial solution,

$$|A| = 0$$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$\begin{vmatrix} 3k-8+3+3 & 3 & 3 \\ 3+3k-8+3 & 3k-8 & 3 \\ 3+3+3k-8 & 3 & 3k-8 \end{vmatrix} = 0$$

$$\begin{vmatrix} 3k-2 & 3 & 3 \\ 3k-2 & 3k-8 & 3 \\ 3k-2 & 3 & 3k-8 \end{vmatrix} = 0$$

$$(3k-2) \begin{vmatrix} 1 & 3 & 3 \\ 1 & 3k-8 & 3 \\ 1 & 3 & 3k-8 \end{vmatrix} = 0$$

$$R_2 - R_1, R_3 - R_1$$

$$(3k-2) \begin{vmatrix} 1 & 3 & 3 \\ 0 & 3k-11 & 0 \\ 0 & 0 & 3k-11 \end{vmatrix} = 0$$

$$(3k-2)(3k-11)^2 = 0$$

$$k = \frac{2}{3}, \frac{11}{3}, \frac{11}{3}$$

T17 : Solution

(b)

Although λ_i^m will be the corresponding eigen values of A^m , x_i^m need not be corresponding eigen vectors.**T18 : Solution**

(d)

Sum of eigen values = $\text{Tr}(A) = -1 + -1 + 3 = 1$ So, $\sum \lambda_i = 1$ Only choice (d) (3, $-1 + 3j$, $-1 - 3j$) gives $\sum \lambda_i = 1$.**T19 : Solution**

(b)

∴ Sum of eigen value of the matrix

$$= \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} \dots \frac{1}{n(n+1)}$$

$$\begin{aligned}
 &= \left[\frac{1}{1} - \frac{1}{2} \right] + \left[\frac{1}{2} - \frac{1}{3} \right] + \left[\frac{1}{3} - \frac{1}{4} \right] \dots \left[\frac{1}{n} - \frac{1}{n+1} \right] \\
 &= \left[1 - \frac{1}{n+1} \right]
 \end{aligned}$$

T20 : Solution

(c)

\therefore For lower triangle matrix eigen value are the element of main diagonal.

\therefore Product of eigen value of the matrix

$$\begin{aligned}
 &= \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \dots \frac{1}{n} \\
 &= \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \dots n} \\
 &= \frac{1}{n!}
 \end{aligned}$$

T21 : Solution

(b)

For a matrix containing complex number, eigen values are real if and only if

$$A = A^\theta = (\bar{A})^T$$

$$A = \begin{bmatrix} 10 & 5+j & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

$$A^\theta = (\bar{A})^T = \begin{bmatrix} 10 & \bar{x} & 4 \\ 5-j & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

By comparing these,

$$x = 5 - j$$

T22 : Solution

(a)

Eigen values of B is are

- (i) $3^2 - 3 = 6$
 - (ii) $2^2 - 2 = 2$
 - (iii) $(-1)^2 + 1 = 2$
- $\therefore |B| = 6 \times 2 \times 2 = 24$

T23 : Solution

(d)

$$\text{Trace of } (I + M + M^2) = \text{tr}(I) + \text{tr}(M) + \text{tr}(M^2)$$

$$\text{Trace of } M = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4$$

$$\text{Trace of } M^2 = 1^2 + (\alpha^1)^2 + (\alpha^2)^2 + (\alpha^3)^2 + (\alpha^4)^2$$

$$\text{Trace of } I = 1 + 1 + 1 + 1 + 1 = 5$$

$$\therefore \alpha = e^{2\pi i/5}$$

α is the fifth root of unity or $x^5 = 1$.

$$\Rightarrow 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$$

$$\begin{aligned} \text{tr}(M^2) &= 1 + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^8 \\ &= 1 + \alpha^2 + \alpha^4 + \alpha + \alpha^3 \\ &= 0 \end{aligned}$$

$$\Rightarrow \text{Trace of } (I + M + M^2) = 5 + 0 + 0 = 5$$

T24 : Solution

The constant term of the polynomial = Determinant of the matrix
 = Product of eigen values

$$\therefore \text{Matrix } I = \begin{bmatrix} 1 & 2 & 6 & 5 \\ -1 & 3 & 2 & -5 \\ 2 & 4 & 12 & 10 \\ 3 & -2 & 1 & -4 \end{bmatrix}$$

$$|I| = \begin{bmatrix} 1 & 2 & 6 & 5 \\ -1 & 3 & 2 & -5 \\ 2 & 4 & 12 & 10 \\ 3 & -2 & 1 & -4 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + R_1; R_3 \rightarrow R_3 - 2R_1$$

$$|I| = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 0 & 5 & 8 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & -2 & 1 & 4 \end{bmatrix}$$

$$|I| = 0$$

Hence, constant term will be zero.



2

Calculus



Detailed Explanation of Try Yourself Questions

T1 : Solution

(b)

$$y = x + \sqrt{x + \sqrt{x + \dots}}$$

$$(y-x)^2 = x + \sqrt{x + \sqrt{x + \dots}}$$

$$y^2 - 2xy + x^2 = x + \sqrt{x + \sqrt{x + \dots}}$$

$$x = 2$$

$$y^2 + 4 - 4y = y$$

$$y^2 - 5y + 4 = 0$$

$$y = 1, 4$$

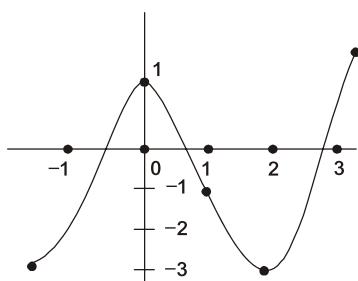
$$y(2) = 2 + \sqrt{2 + \sqrt{2 + \dots}} > 2$$

∴

$$y(2) = 4 \text{ only}$$

T2 : Solution

(b)



$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x$$

$$f'(x) = 0$$

$$3x^2 - 6x = 0$$

$$\begin{aligned}
 3x(x - 2) &= 0 \\
 x &= 0, 2 \\
 f''(x) &= 6x - 6 \\
 \text{At} &\quad x = 0 \quad f''(0) = -6 \text{ maxima} \\
 &\quad x = 2 \quad f''(2) = 6 \text{ minima}
 \end{aligned}$$

T3 : Solution

$$\begin{aligned}
 |4x - 7| &= 5 \\
 4x - 7 &= -5 \\
 4x &= 2 \\
 x &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 2|x| - |-x| & \\
 2\left|\frac{1}{2}\right| - \left|-\frac{1}{2}\right| & \\
 2\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right) &= \frac{1}{2} \\
 4x - 7 &= 5 \\
 4x &= 12 \\
 x &= 3 \\
 2|x| - |-x| & \\
 2|3| - |-3| & \\
 2(3) - 3 &= 3
 \end{aligned}$$

T4 : Solution

$$u_x = \begin{vmatrix} 2x & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\begin{aligned}
 \text{Similarly } u_y &= 0 \quad u_z = 0 \\
 \therefore u_x + u_y + u_z &= 0
 \end{aligned}$$

T5 : Solution

(c)

$$\begin{aligned}
 y &= |2 - 3x| = 2 - 3x \quad 2 - 3x \geq 0 \\
 &= 3x - 2 \quad 2 - 3x < 0
 \end{aligned}$$

$$\text{Therefore, } y = 2 - 3x \quad x \leq \frac{2}{3}$$

$$= 3x - 2 \quad x > \frac{2}{3}$$

Since $2 - 3x$ and $3x - 2$ are polynomials, these are continuous at all points. The only concern is at $x = 2/3$

Left limit at $x = \frac{2}{3}$ is $2 - 3 \times \frac{2}{3} = 0$.

Right limit at $x = \frac{2}{3}$ is $3 \times \frac{2}{3} - 2 = 0$.

$$f\left(\frac{2}{3}\right) = 2 - 3 \times \frac{2}{3} = 0$$

Since, Left limit = Right limit = $f\left(\frac{2}{3}\right)$,

Function is continuous at $\frac{2}{3}$.

y is therefore continuous $\forall x \in R$

Now since $2 - 3x$ and $3x - 2$ are polynomials, they are differentiable.

only concern is at $x = \frac{2}{3}$.

Now, at $x = \frac{2}{3}$, LD = Left derivative = -3

$$\begin{aligned} RD &= \text{Right derivative} = +3 \\ LD &\neq RD \end{aligned}$$

\therefore The function y is not differentiable at $x = \frac{2}{3}$

So, we can say that y is differentiable $\forall x \in R$, except at $x = \frac{2}{3}$.

T6 : Solution

(c)

If $f(x)$ is continuous at $x = \frac{\pi}{2}$

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\lambda \cos x}{\frac{\pi}{2} - x} = f\left(\frac{\pi}{2}\right) = 1 \quad \dots(i)$$

Since the limit is in form of $\frac{0}{0}$, we can use L' hospitals rule on LHS of equation (i) and get

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{-\lambda \sin x}{-1} = 1$$

$$\Rightarrow \lambda \sin \frac{\pi}{2} = 1$$

$$\Rightarrow \lambda = 1$$

T7 : Solution

(c)

$$\begin{aligned} |x| &= x & x \geq 0 \\ &= -x & x < 0 \end{aligned}$$

at $x = 0$ left limit = 0Right limit = $-0 = 0$

$f(0) = 0$

Since left limit = Right limit = $f(0)$ So $|x|$ is continuous at $x = 0$

Now,

LD = Left derivative (at $x = 0$) = -1RD = Right derivative (at $x = 0$) = +1LD \neq RDSo $|x|$ is not differentiable at $x = 0$ So $|x|$ is continuous and non-differentiable at $x = 0$ **T8 : Solution**

(d)

 $f(x)$ is continuous at any point

if $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$

T9 : Solution

(b)

 P : If $f(x)$ is continuous at $x = x_0$, then it is also differentiable at $x = x_0$ Q : If $f(x)$ is continuous at $x = x_0$, then it may or may not be derivable at $x = x_0$ R : If $f(x)$ is differentiable at $x = x_0$, then it is also continuous at $x = x_0$ P is false Q is true R is true

Option (b) is correct

T10 : Solution

(a)

$$\begin{cases} 2, & \text{if } x = 3 \\ x - 1, & \text{if } x > 3 \\ \frac{x+3}{3}, & \text{if } x < 3 \end{cases}$$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} \frac{x+3}{3} = 2$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} x - 1 = 2$$

Also, $f(3) = 2$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3)$$

So it is continuous at $x = 3$

option (a) is correct.

T11 : Solution

At $x = 0$

$$\frac{\sin 0}{e^0 \cdot 0} = \frac{0}{0}$$

So applying L'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{\cos x}{xe^x + e^x} = \frac{1}{0+1} = 1$$

T12 : Solution

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{1 - \cos x}$$

Applying L' Hospital's rule

$$\lim_{x \rightarrow 0} \frac{\frac{d}{dx}(x - \sin x)}{\frac{d}{dx}(1 - \cos x)} = \lim_{x \rightarrow 0} \frac{(1 - \cos x)}{\sin x}$$

(It is still of $\frac{0}{0}$ form)

Again applying L' Hospital's rule

$$\lim_{x \rightarrow 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(\sin x)} = \lim_{x \rightarrow 0} \frac{\sin x}{\cos x} = \frac{0}{1} = 0$$

T13 : Solution

$$\lim_{x \rightarrow 0} \frac{(e^{2x} - 1)}{\sin 4x}, \text{ it is of } \left(\frac{0}{0}\right) \text{ from}$$

Applying L' Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{2e^{2x}}{4\cos 4x} = \frac{2 \times 1}{4 \times 1} = \frac{1}{2}$$

T14 : Solution

$$\lim_{x \rightarrow 0} \left(\frac{-\sin x}{2\sin x + \cos x} \right) = \lim_{x \rightarrow 0} \left(\frac{-\sin 0}{2\sin 0 + \cos 0} \right) = \frac{0}{1} = 0$$

(Note: Since the function is not evaluating to 0/0 not need to use L' Hospital's rule)

T15 : Solution

(d)

- (a) $f(x) = \tan(\pi x)$ is not continuous at $x = \frac{1}{2}$.
- (b) $f(x)$ is continuous at $x = \frac{1}{2}$ but not differentiable.
- (c) $f(0) = 0$
 $f(1) = (1)^2 = 1$
 and $f(0) \neq f(1)$
- (d) $f(x) = \sqrt{x(1-x)}$

$f(x)$ is continuous in $[0, 1]$ and differentiable in $(0, 1)$ and $f(0) = f(1) = 1$.
 So, all these conditions are satisfied. Hence, Rolle's theorem is valid.

T16 : Solution

(d)

Maclaurin series expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots$$

$$\text{at } x = 1, \quad e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \dots$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

T17 : Solution

(1)

Maclaurin series : Expansion of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$$

$$\text{at } x = \frac{\pi}{2} \Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots \right) = \sin\left(\frac{\pi}{2}\right) = 1$$

T18 : Solution

(c)

We need absolute maximum of

$$f(x) = x^3 - 9x^2 + 24x + 5 \text{ in the interval } [1, 6]$$

First find local maximum if any by putting $f'(x) = 0$.

$$\text{i.e. } f'(x) = 3x^2 - 18x + 24 = 0$$

$$\text{i.e. } x^2 - 6x + 8 = 0$$

$$x = 2, 4$$

Now

$$f''(x) = 6x - 18$$

$$f''(2) = 12 - 18 = -6 < 0$$

and

$$f''(4) = 24 - 18 = +6 > 0$$

(So $x = 2$ is a point of local maximum)

(So $x = 4$ is a point of local minimum)

Now tabulate the values of f at end point of interval and at local maximum point, to find absolute maximum in given range, as shown below:

x	$f(x)$
1	21
2	25
6	41

Clearly the absolute maxima is at $x = 6$
and absolute maximum value is 41.

T19 : Solution

(d)

$f(x)$ has a local minimum at $x = x_0$

$$\text{if } f'(x_0) = 0$$

$$\text{and } f''(x_0) > 0$$

T20 : Solution

$$f(x) = x^3 - 9x^2 + 24x + 5$$

$$f'(x) = 3x^2 + 18x + 24$$

$$f''(x) = 6x - 18$$

$$f'(x) = 0$$

$$3x^2 + 18x + 24 = 0$$

$$x^2 - 6x + 8 = 0$$

$$x = 2, 4$$

$$x = 2, \quad f''(2) = -6$$

\therefore Relative maxima

$$x = 4, \quad f''(4) = 6 \text{ relative minima}$$

\therefore Maximum value of $f(x) = f(2) = 25$.

T21 : Solution

(6)

$$\begin{aligned}
 f(x) &= 2x^3 - 9x^2 + 12x - 3 \\
 f'(x) &= 6x^2 - 18x + 12 \\
 f''(x) &= 12x - 18 \\
 f'(x) &= 0 \\
 x^2 - 3x + 2 &= 0 \\
 x = 1, x = 2 & \\
 f''(1) = -6; f''(2) = 6 &
 \end{aligned}$$

$x = 1$ is local maxima point.

$$\begin{aligned}
 \text{Global maxima of } f(x) &= \text{Max}\{f(0), f(1), f(3)\} \\
 &= \{-3, 2, 6\} = 6
 \end{aligned}$$

T22 : Solution

(10)

$$\begin{aligned}
 \text{Given : } 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 & \\
 &= 5\cos\theta + 3\cos\theta\cos\frac{\pi}{3} - 3\sin\theta\sin\frac{\pi}{3} + 3 \\
 &= 5\cos\theta + \frac{3}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3 \\
 &= \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3
 \end{aligned}$$

\therefore The maximum value of $a \sin \theta + b \cos \theta$ is $\sqrt{a^2 + b^2}$.

\therefore Maximum value of $\frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$ is

$$\begin{aligned}
 &= \sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{-3\sqrt{3}}{2}\right)^2} \\
 &= \sqrt{\frac{169}{4} + \frac{27}{4}} \\
 &= \sqrt{49} = 7
 \end{aligned}$$

$$\begin{aligned}
 \text{So maximum value of } \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3 & \text{ is} \\
 &= 7 + 3 = 10
 \end{aligned}$$

T23 : Solution

(d)

Given : $f(x, y) = \frac{ax^2 + by^2}{xy} = a\left(\frac{x}{y}\right) + b\left(\frac{y}{x}\right)$

$$\frac{\partial f}{\partial x} = \frac{a}{y} - \frac{by}{x^2}; \quad \frac{\partial f}{\partial y} = -\frac{ax}{y^2} + \frac{b}{x}$$

$$\left. \frac{a}{y} - \frac{by}{x^2} \right|_{\substack{x=1 \\ y=2}} = \left. -\frac{ax}{y^2} + \frac{b}{x} \right|_{\substack{x=1 \\ y=2}}$$

$$\frac{a}{2} - \frac{2b}{1} = -\frac{a}{4} + b$$

$$\frac{3a}{4} = 3b$$

$$a = 4b$$

T24 : Solution

(d)

$$x^a y^b = (x + y)^{a+b}$$

Differentiating w.r.t x

$$\begin{aligned} &= ax^{a-1}y^b + bx^a y^{b-1} \frac{dy}{dx} \\ &= (a+b)(x+y)^{(a+b-1)} \left(1 + \frac{dy}{dx} \right) \end{aligned}$$

$$\frac{ax^a y^b}{x} + \frac{bx^a y^b}{y} \times \frac{dy}{dx} = (a+b) \frac{(x+y)^{a+b}}{x+y} \left(1 + \frac{dy}{dx} \right)$$

$$ax^a y^b \left[\frac{a}{x} + \frac{by}{yx} \right] = (a+b) \frac{(x+y)^{a+b}}{x+y} \left(1 + \frac{dy}{dx} \right)$$

$$\frac{a}{x} - \frac{a+b}{x+y} = \left(\frac{a+b}{x+y} - \frac{b}{y} \right) \frac{dy}{dx}$$

$$\frac{ax + ay - ax - bx}{(x)(x+y)} = \left(\frac{ay + by - bx - by}{(y)(x+y)} \right) \frac{dy}{dx}$$

$$\frac{ay - bx}{x(x+y)} = \left(\frac{ay - bx}{y(x+y)} \right) \frac{dy}{dx}$$

$$= \frac{y}{x}$$

T25 : Solution

(c)

$$w = f(x, y)$$

By chain rule,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \times \frac{dx}{dt} + \frac{\partial w}{\partial y} \times \frac{dy}{dt}$$

T26 : Solution

(a)

$$Z = f(x, y)$$

$$x = e^u + e^{-v}$$

$$y = e^{-u} - e^v$$

$$\begin{aligned} Z_u &= \frac{\partial Z}{\partial u} = \frac{\partial Z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial Z}{\partial y} \cdot \frac{\partial y}{\partial u} \\ &= \frac{\partial Z}{\partial x} \cdot e^u + \frac{\partial Z}{\partial y} \cdot (-e^{-u}) \end{aligned}$$

$$= e^u Z_x - e^{-u} Z_y$$

$$\begin{aligned} Z_v &= \frac{\partial Z}{\partial v} = \frac{\partial Z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial Z}{\partial y} \cdot \frac{\partial y}{\partial v} \\ &= \frac{\partial Z}{\partial x} (-e^{-v}) + \frac{\partial Z}{\partial y} (-e^v) \end{aligned}$$

$$= -e^{-v} Z_x - e^v Z_y$$

$$\begin{aligned} Z_u - Z_v &= e^u \cdot Z_x - e^{-v} Z_y - (-e^{-v} Z_x - e^v Z_y) \\ &= x Z_x - y Z_y \end{aligned}$$

T27 : Solution

(c)

$$u = x^n f_1\left(\frac{x}{y}\right) + y^{-n} f_2\left(\frac{x}{y}\right)$$

Since u is homogeneous of degree n .

$$\text{So } x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n u \quad \dots(i)$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (n)(n-1)u \quad \dots(ii)$$

Adding (i) and (ii)

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n u + (n^2 - n)u = n^2 u$$

T29 : Solution

(b)

$$\begin{aligned}\int_0^\infty \frac{1}{1+x^2} dx &= \left[\tan^{-1} x \right]_0^\infty \\ &= \tan^{-1} \infty - \tan^{-1} 0 = \frac{\pi}{2}\end{aligned}$$

and

$$L(\sin x) = \frac{1}{s^2 + 1}$$

$$\begin{aligned}\Rightarrow L\left(\frac{\sin x}{x}\right) &= \int_s^\infty \frac{1}{s^2 + 1} ds \\ &= \left[\tan^{-1} s \right]_s^\infty \\ &= \tan^{-1} \infty - \tan^{-1}(s) = \cot^{-1}(s)\end{aligned}$$

(Using “division by x ”)

$$\Rightarrow \int_0^\infty e^{-sx} \frac{\sin x}{x} dx = \cot^{-1}(s) \quad (\text{Using definition of Laplace transform})$$

Put $s = 0$,

$$\text{we get } \int_0^\infty \frac{\sin x}{x} dx = \cot^{-1}(0) = \frac{\pi}{2}$$

$$\frac{1}{1+x^2} dx + \int_0^\infty \frac{\sin x}{x} dx = \pi$$

T30 : Solution

(a)

Let,

$$\sin^{-1} x = t$$

$$\frac{dx}{\sqrt{1-x^2}} = dt$$

$$I = \int_0^{\pi/2} t^2 dt = \left[\frac{t^3}{3} \right]_0^{\pi/2} = \frac{\pi^3}{24}$$

T31 : Solution

(b)

$$\begin{aligned}I &= \int_0^{\pi/2} \log(\sin x) dx \\ &= \int_0^{\pi/2} \log\left(\sin\left(\frac{\pi}{2} - x\right)\right) dx \\ I &= \int_0^{\pi/2} \log \cos x dx \\ 2I &= \int_0^{\pi/2} \log \sin x dx + \int_0^{\pi/2} \log \cos x dx\end{aligned}$$

$$\begin{aligned}
 &= \int_0^{\pi/2} \log \sin x \cos x dx \\
 &= \int_0^{\pi/2} \log 2 \sin x \cos x dx - \int_0^{\pi/2} \log 2 dx \\
 &= \int_0^{\pi/2} \log \sin 2x dx - \int_0^{\pi/2} \log 2 dx \\
 &= \frac{1}{2} \int_0^{\pi} \log \sin u du - \frac{\pi}{2} \log 2 \\
 &= \int_0^{\pi/2} \log \sin u du - \frac{\pi}{2} \log 2 \\
 &= I - \frac{\pi}{2} \log 2 \\
 I &= -\frac{\pi}{2} \log 2
 \end{aligned}$$

T32 : Solution

(b)

$$I = \int_0^\infty e^{-x^2/2} dx \text{ or } I = \int_{-\infty}^\infty e^{-y^2/2} dy$$

$$\begin{aligned}
 I^2 &= \int_{-\infty}^\infty e^{-x^2/2} dx \int_{-\infty}^\infty e^{-y^2/2} dy \\
 &= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)/2} dx dy
 \end{aligned}$$

$$r = \sqrt{x^2 + y^2}$$

$$\begin{aligned}
 I^2 &= \int_0^{2\pi} \int_0^\infty e^{-r^2/2} r dr d\theta \\
 &= \int_0^{2\pi} d\theta \int_0^\infty r e^{-r^2/2} dr \\
 &= 2\pi \int_0^\infty e^{-t} dt
 \end{aligned}$$

Let $\frac{r^2}{2} = t, r dr = dt$

$$= 2\pi \left. \frac{e^{-t}}{-1} \right|_0^\infty = 2\pi$$

$$I^2 = 2\pi$$

$$\Rightarrow I = \sqrt{2\pi}$$

T33 : Solution

(b)

$$I = \int_0^n \frac{1 - \left(\frac{x}{n}\right)^n}{n-x} dx = \int_0^n \frac{1 - \left(\frac{x}{n}\right)^n}{1 - \left(\frac{x}{n}\right)} \frac{dx}{n}$$

$$\text{Let, } \frac{x}{n} = t \quad x \rightarrow 0 \quad t \rightarrow 0 \\ x \rightarrow n \quad t \rightarrow 1$$

$$\frac{dx}{n} = dt$$

$$\Rightarrow I = \int_0^1 \frac{1-t^n}{1-t} dt$$

$$\frac{1-t^n}{1-t} = \sum_{r=1}^n t^{r-1}$$

$$I = \int_0^1 \sum_{r=1}^n t^{r-1} dt$$

Changing order of summation and integral

$$I = \sum_{r=1}^n \int_0^1 t^{r-1} dt = \sum_{r=1}^n \frac{t^r}{r} \Big|_0^1 = \sum_{r=1}^n \frac{1}{r}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

T34 : Solution

$$I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx$$

$$I = \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2} - x\right) - \sin\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$$

$$= - \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx$$

$$I = -I$$

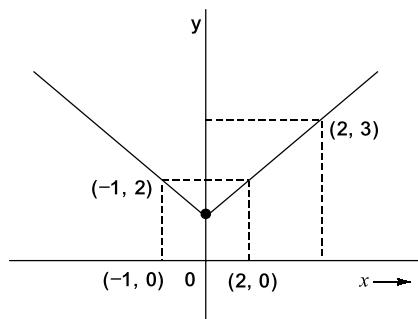
$$2I = 0$$

$$\Rightarrow I = 0$$

T36 : Solution

$$\int_{-1}^2 (1+|x|) dx$$

$$y = 1 + |x|$$



$$\int_{-1}^2 (1+|x|) dx = \text{Area under the curve}$$

$$\begin{aligned} &= \int_{-1}^2 y \cdot dx = \frac{1}{2} [(1+2) \times 1 + (1+3) \times 2] \\ &= 5.5 \end{aligned}$$

T37 : Solution

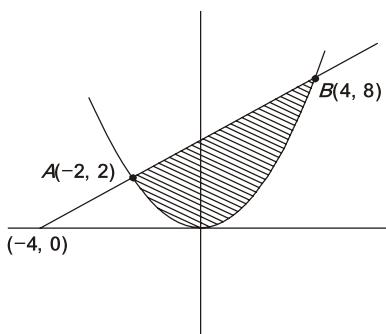
$$\int_0^1 \frac{1}{\sqrt{1-x^2}} dx = (\sin^{-1} x) \Big|_0^1 = \sin^{-1} 1 = \frac{\pi}{2}$$

T38 : Solution

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx &= 2 \int_0^{\infty} \frac{1}{1+x^2} dx \\ &= 2 \left(\tan^{-1} x \right) \Big|_0^{\infty} = 2 \tan^{-1} \infty = \pi \end{aligned}$$

T39 : Solution

(b)



Point of intersection

$$x = \frac{x^2}{2} - 4$$

$$2x = x^2 - 8$$

$$x^2 - 2x - 8 = 0$$

$$(x - 4)(x + 2) = 0$$

$$x = -2, x = 4$$

Point of intersection are (-2, 2), (4, 8)

$$\begin{aligned} \text{Area} &= \int_{-2}^4 \int_{y=\frac{x^2}{2}}^{y=x+4} dy dx \\ &= \int_{-2}^4 \left[x + 4 - \frac{x^2}{2} \right] dx \\ &= \left. \frac{x^2}{2} + 4x - \frac{x^3}{6} \right|_{-2}^4 \\ &= \frac{1}{2}(16 - 4) + 4(4 + 2) - \frac{1}{6}(64 + 8) \\ &= 6 + 24 - 12 = 18 \end{aligned}$$

T40 : Solution

$$\begin{aligned} &\int_1^2 \left[\int_0^x \frac{1}{x^2 + y^2} dy \right] dx \\ &= \int_1^2 \left[\frac{1}{x} \tan^{-1} \left(\frac{y}{x} \right) \right]_0^x dx \\ &= \int_1^2 \frac{\pi}{4} \cdot \frac{1}{x} dx = \frac{\pi}{4} (\log x) \Big|_1^2 = \frac{\pi}{4} \log 2 \end{aligned}$$

T41 : Solution

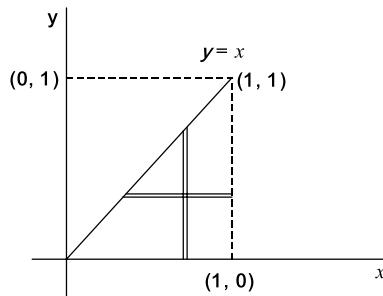
$$\begin{aligned} &\int_0^\infty \left[\int_0^y y \cdot e^{-y} \cdot e^{-x} dx \right] dy \\ &\int_0^\infty y e^y \left(\frac{e^{-x}}{-1} \right) \Big|_0^y dy = \int_0^\infty y e^{-y} \left(\frac{e^{-y} - 1}{-1} \right) dy \\ &= \int_0^\infty y e^{-y} (1 - e^{-y}) dy = \frac{3}{4} \end{aligned}$$

T42 : Solution

(d)

$$I = \int_0^1 \int_y^1 y \sqrt{1+x^3} dx dy$$

$$I = \int_0^1 \int_0^x y \sqrt{1+x^3} dx dy$$



$$I = \int_0^1 \frac{y^2}{2} \Big|_0^x \sqrt{1+x^3} dx = \frac{1}{2} \int_0^1 x^2 \sqrt{1+x^3} dx$$

Let

$$1 + x^3 = t$$

$$3x^2 dx = dt$$

$$= \frac{1}{2} \times \frac{1}{3} \int_1^2 \sqrt{t} dt$$

$$= \frac{1}{6} \frac{t^{3/2}}{3/2} \Big|_1^2$$

$$= \frac{1}{9} [2^{3/2} - 1^{3/2}] = \frac{1}{9} (2\sqrt{2} - 1)$$

T43 : Solution

(a)

$$\phi(r, \theta) = J\left(\frac{x, y}{r, \theta}\right) = \begin{vmatrix} x_r & y_r \\ x_\theta & y_\theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta
= r$$



3

Vector Calculus



Detailed Explanation of Try Yourself Questions

T1 : Solution

(b)

$$f = x^2 + 3y^2 + 2z^2$$

$$\begin{aligned}\Delta f &= \text{grad } f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} \\ &= i(2x) + j(6y) + k(4z)\end{aligned}$$

The gradient at $P(1, 2, -1)$ is

$$\begin{aligned}&= i(2 \times 1) + j(6 \times 2) + k(4 \times -1) \\ &= 2i + 12j - 4k\end{aligned}$$

T2 : Solution

(a)

$$\phi = xy^2 + yz^2 + zx^2$$

$$\nabla \phi = \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z}$$

$$= \bar{i}(y^2 + 2xz) + \bar{j}(2xy + z^2) + \bar{k}(2yz + x^2)$$

$$\begin{aligned}\nabla \phi_{(2, -1, 1)} &= \bar{i}(1+4) + \bar{j}(-4+1) + \bar{k}(-2+4) \\ &= 5\bar{i} - 3\bar{j} + 2\bar{k}\end{aligned}$$

$$\bar{P} = \bar{i} + 2\bar{j} + 2\bar{k}$$

$$|\bar{P}| = \sqrt{1+4+4} = 3$$

The directional derivative of $\phi(x, y, z)$ at $(2, -1, 1)$ in the direction of \bar{P} is $\nabla \phi_{\text{at } P} \cdot \frac{\bar{P}}{|\bar{P}|}$

$$\begin{aligned}
 &= (5\bar{i} - 3\bar{j} + 2\bar{k}) \cdot \left(\frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3} \right) \\
 &= \frac{5 - 6 + 4}{3} = 1
 \end{aligned}$$

T3 : Solution

(d)

$$\operatorname{div}\{(x-y)\hat{i} + (y-x)\hat{j} + (x+y+z)\hat{k}\} = \frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}(y-x) + \frac{\partial}{\partial z}(x+y+z) = 3$$

T4 : Solution

(d)

$$\begin{aligned}
 \nabla \cdot \vec{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
 \nabla \cdot \vec{A} &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 \\
 \nabla \cdot \vec{A} &= 3
 \end{aligned}$$

T5 : Solution

(a)

$$\vec{F} = x^2 z^2 \vec{i} - 2xy^2 z \vec{j} + 2y^2 z^3 \vec{k}$$

$$\begin{aligned}
 \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z^2 & -2xy^2 z^2 & 2y^2 z^3 \end{vmatrix} \\
 &= \vec{i} \left[\frac{\partial}{\partial y}(2y^2 z^3) + \frac{\partial}{\partial z}(2xy^2 z) \right] \\
 &\quad - \vec{j} \left[\frac{\partial}{\partial y}(2y^2 z^3) - \frac{\partial}{\partial z}(x^2 z^2) \right] \\
 &\quad + \vec{k} \left[\frac{\partial}{\partial x}(-2xy^2 z) - \frac{\partial}{\partial y}(x^2 z^2) \right]
 \end{aligned}$$

$$\begin{aligned}
 \nabla \times \vec{F} &= \vec{i}[4yz^3 + 2xy^2] - \vec{j}[2zx^2] \\
 &\quad + \vec{k}[-2y^2 z - 0] \\
 &= (4yz^3 + 2xy^2)\vec{i} - (2x^2 z)\vec{j} - (2y^2 z)\vec{k}
 \end{aligned}$$

T6 : Solution

$A(0, 2, 1)$ and $B(4, 1, -1)$

The equation of the line AB is

$$\frac{x-0}{4-0} = \frac{y-2}{1-2} = \frac{z-1}{-1-1} = t \text{ say}$$

$$x = 4t ; y = -t + 2 ; z = -2t + 1$$

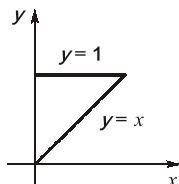
$$dx = 4dt ; dy = -dt ; dz = -2dt$$

t varies from 0 to 1

$$\begin{aligned} I &= \int_0^1 2(-2t+1) 4dt + 2(-t+2)(-dt) + 2(4t)(-2dt) \\ &= \int_0^1 (-16t+8+2t-4-16t)dt \\ &= \int_0^1 (-30t+4)dt \\ &= \left[-30 \frac{t^2}{2} + 4t \right]_0^1 = -15 + 4 = -11 \end{aligned}$$

T7 : Solution

(b)



$$F_1 = x^2 + 2\sin^2 x \cos x \quad F_2 = 4x + y^2$$

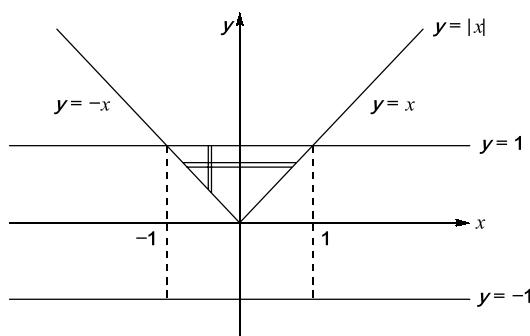
$$\frac{\partial F_1}{\partial y} = 0 \quad \frac{\partial F_2}{\partial x} = 4$$

$$\begin{aligned} \iint_C F_1 dx + F_2 dy &= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy \\ &= \iint_R (4 - 0) dx dy = 4 \cdot \left(\frac{1}{2} \right) = 2 \end{aligned}$$

T8 : Solution

$$I = \left[\left\{ xy^2 + 2y + \sin(c^x) \right\} dx + \left\{ x^2 y + \cos(e^y) \right\} dy \right]$$

$$c' = \{x, y \in R^2 \mid \max\{|x|, |y|\} = 1\}$$



$$\begin{aligned}
 \int \phi dx + \psi dy &= \iint \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx dy \\
 \frac{\partial \psi}{\partial x} = 2xy &= \frac{\partial \phi}{\partial y} = 2xy + 2 \\
 &= \iint (2xy - 2xy - 2) dx dy \\
 &= \int_0^1 \int_{-y}^y -2 dx dy \\
 &= -2 \int_0^1 x \Big|_{-y}^y dy = -2 \int_0^1 2y dy = -2 \times 2 \frac{y^2}{2} \Big|_0^1 \\
 &= -2
 \end{aligned}$$

T9 : Solution

$$I = \int_C e^x dx + 2y dy - dz$$

Stoke's Theorem

$$\int_C F \cdot dr = \iint_R (\nabla \times F) \cdot \hat{n} ds$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k}$$

 Hence, $I = 0$.

T10 : Solution

(a)

$$\bar{F} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$\nabla \cdot \bar{F} = 3$$

By Gauss divergence theorem

$$\int_s \bar{F} \cdot \hat{n} ds = \int_v \nabla \cdot \bar{F} dv = \int_v 3 dv = 3 V$$

where V is volume of $x^2 + y^2 + z^2 = 4$

$$= 3 \left(\frac{32\pi}{3} \right) = 32\pi$$



4

Differential Equations



Detailed Explanation of Try Yourself Questions

T1 : Solution

$$(D^4 - D^2)y = 0$$

Auxiliary equation is

$$m^4 - m^2 = 0$$
$$m = 0, 0 \quad m = \pm 1$$

Complimentary function is

$$(C_1 + C_2x) + C_3e^{-x} + C_4e^x = C_1 + C_2x + C_3e^{-x} + C_4e^x$$

Set of independent = {1, x, e^{-x}, e^x}

T2 : Solution

(b)

$$\frac{dy}{dx} = (4x + y + 1)^2$$

Let,

$$4x + y + 1 = t$$

Differentiating w.r.t. x

$$4 + \frac{dy}{dx} = \frac{dt}{dx}$$
$$\Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 4$$
$$\frac{dt}{dx} - 4 = t^2$$
$$\frac{dt}{dx} = t^2 + 4$$
$$\int \frac{dt}{t^2 + 4} = \int dx$$

$$\begin{aligned}\frac{1}{2} \tan^{-1}\left(\frac{t}{2}\right) &= x + c \\ \frac{1}{2} \tan^{-1}\left(\frac{4x+y+1}{2}\right) &= x + c\end{aligned}$$

T3 : Solution

(a)

Let the number of bacteria at time $t = n$

$$\begin{aligned}\frac{dn}{dt} &= n \\ \frac{dn}{dt} &= kn \\ \int_{n_0}^{2n_0} \frac{dn}{n} &= \int_0^2 k dt \\ \ln n|_{n_0}^{2n_0} &= kT|_0^2 \\ 2k &= \ln 2 \\ k &= \frac{1}{2} \ln 2 \\ \Rightarrow \ln\left(\frac{n}{n_0}\right) &= \frac{1}{2} \ln 2(t - 0) \\ \ln\left(\frac{3n_0}{n_0}\right) &= \frac{1}{2} \ln 2(t - 0) \\ \ln 3 &= \frac{1}{2} \ln 2t \\ \Rightarrow t &= 2 \frac{\ln 3}{\ln 2}\end{aligned}$$

T4 : Solution

(a)

Let the rate of cooling is $\frac{dT}{dt}$ where T is temperature.

$$\frac{dT}{dt} \propto (T - T_o)$$

where, T is temperature of the body and T_o is temperature of the air.

$$\begin{aligned}\int_{100^\circ}^{75^\circ} \frac{dT}{T - T_o} &= \int_0^1 -kt \\ \ln(T - T_o)|_{100^\circ}^{75^\circ} &= -kt|_0^1 \\ \ln\left(\frac{75^\circ - 25^\circ}{100^\circ - 25^\circ}\right) &= -kt|_0^1\end{aligned}$$

$$\ln\left(\frac{50^\circ}{75^\circ}\right) = -k$$

$$\Rightarrow k = \ln\left(\frac{3}{2}\right)$$

After 3 minutes, temperature = T

$$\ln(T - T_0)|_{100^\circ}^T = -\ln\left(\frac{3}{2}\right) \cdot t \Big|_0^3$$

$$\ln\left(\frac{T - 25^\circ}{100^\circ - 25^\circ}\right) = -\ln\frac{3}{2} \times 3$$

$$T = 47.22^\circ\text{C}$$

From here,

T5 : Solution

(d)

Let the amount of radium at time t is x grams.

$$\frac{dx}{dt} \propto -x$$

$$\frac{dx}{dt} = kx$$

$$x = x_0 e^{-kt}$$

\Rightarrow Let initial amount was 100%. 5% disappeared in 50 years

\Rightarrow Remaining amount will be 95%

$$95 = 100 e^{-k \cdot 50}$$

$$e^{-k \times 50} = \frac{95}{100}$$

After 100 years

$$x = 100 \times e^{-k \cdot 100}$$

$$x = 100 \times \left(\frac{95}{100}\right)^2$$

$$= \frac{9025}{100} = 90.25\%$$

T6 : Solution

Differential equation $Mdx + Ndy = 0$ is exact.

$$\text{If } \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

\therefore

$$M = 27x^2 + Ky \cos x$$

$$N = 2 \sin x - 27y^3$$

$$\frac{\partial N}{\partial x} = 2 \cos x$$

$$\frac{\partial M}{\partial y} = K \cos x$$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

$$2 \cos x = K \cos x$$

$$K = 2$$

Hence,

T7 : Solution

(b)

$$\frac{dy}{dx} + \tan x \tan y = \cos x \sec y$$

$$\frac{dy}{dx} + \frac{\sin x \sin y}{\cos x \cos y} = \cos x$$

$$\cos y \frac{dy}{dx} + \frac{\sin x \sin y}{\cos x} = \cos x$$

Let,

$$\cos y \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + \delta \tan x \cdot t = \cos x$$

$$\text{I.F.} = e^{\int \tan x dx} = e^{\ln \sec x}$$

$$= \sec x$$

$$y(\sec x) = \cos x \times \frac{1}{\sec x} dx$$

$$\sin y \cdot \sec x = x + c$$

$$\sin y = (x + c) \cos x$$

T8 : Solution

(d)

The equation can be re-written as

$$\frac{dy}{dx} \cos(x+y) + \frac{\sin(x+y)}{x} - e^x + \cos(x+y) = 0$$

$$\left[\frac{\sin(x+y)}{x} - e^x + \cos(x+y) \right] dx + \cos(x+y) dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\cos(x+y)}{x} - \sin(x+y)$$

$$\frac{\partial N}{\partial x} = -\sin(x+y)$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{\frac{\cos(x+y)}{x} - \sin(x+y) - (-\sin(x+y))}{\cos(x+y)}$$

$$= \frac{1}{x} \quad [\text{exclusive function of } x]$$

$$\begin{aligned}\text{I.F.} &= e^{\int \frac{1}{x} dx} = e^{\ln x} \\ &= x\end{aligned}$$

⇒ Differential equation becomes

$$[\sin(x+y) - xe^x + x \cos(x+y)]dx + x \cos(x+y)dy = 0$$

$$\int (\text{terms contain 'x' only})dx + \int x \cos(x+y)dy = 0$$

$$\int -xe^x dx + \int x \cos(x+y)dy = 0$$

$$\begin{aligned}\int x \cos(x+y)dy &= \int xe^x dx \\ x \sin(x+y) &= xe^x - e^x + C\end{aligned}$$

$$x(\sin(x+y) - e^x) + e^x = \text{Constant.}$$

T9 : Solution

(a)

The differential equation

$$(t^2 - 81) \frac{dy}{dt} + 5ty = \sin t$$

$$\frac{dy}{dt} + \left(\frac{5t}{t^2 - 81} \right) y = \frac{\sin t}{t^2 - 81}$$

$$P = \frac{5t}{t^2 - 81}$$

$$Q = \frac{\sin t}{t^2 - 81}$$

$$I.F. = e^{\int P dt} = e^{\int \frac{5t}{t^2 - 81} dt}$$

$$= e^{5/2 \int \frac{2t}{t^2 - 81} dt}$$

$$= e^{5/2 \ln(t^2 - 81)}$$

$$= (t^2 - 81)^{5/2}$$

Solution is

$$y(t^2 - 81)^{5/2} = \int \frac{\sin t}{(t^2 - 81)} \cdot (t^2 - 81)^{5/2} dt$$

$$y = \int \frac{\sin t}{(t^2 - 81)} dt + \frac{C}{(t^2 - 81)^{5/2}}$$

The solution exists for $t \neq \pm 9$. Hence only (a) option is not involving ± 9 .

T10 : Solution

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^2 - 2D + 2} \log 2 \\
 &= \frac{1}{D^2 - 2D + 2} (\log 2) \cdot e^{0x} \\
 &= \frac{\log 2}{2}
 \end{aligned}$$

Put $D = 0$

T11 : Solution

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^3 - D^2 - D} (-e^x) \\
 &= \frac{1}{1-1-1} (-e^x) = e^x
 \end{aligned}$$

Put $D = 1$

T12 : Solution

(d)

$$\begin{aligned}
 x^2 \frac{dy}{dx} + 2xy &= \frac{2\log x}{x} \\
 \frac{dy}{dx} + \frac{2}{x}y &= \frac{2\log x}{x^3} \\
 \text{I.F.} &= e^{\int \frac{2}{x} dx} = 2^{2\ln x} = x^2
 \end{aligned}$$

Solution of differential equation is

$$\begin{aligned}
 y \cdot \text{I.F.} &= \int Q(x) \cdot \text{I.F.} dx \\
 y \cdot x^2 &= \int \frac{2\log x}{x^3} \cdot x^2 dx \\
 y \cdot x^2 &= 2 \int \frac{\log x}{x} dx \\
 y \cdot x^2 &= 2 \int t dt \quad \left[\text{Let } \log x = t, \frac{1}{x} dx = dt \right] \\
 y \cdot x^2 &= 2 \frac{t^2}{2} + C \\
 y \cdot x^2 &= (\log x)^2 + C \\
 y(1) &= 0 \\
 0 \times 1 &= [\log(1)^2] + C \Rightarrow C = 0 \\
 y(e) &= ? \\
 y \cdot e^2 &= [\log e]^2 = 1 \\
 y &= \frac{1}{e^2}
 \end{aligned}$$

T13 : Solution

(b)

$$(x^3 D^3 + 2x^2 D^2 + 2)y = 0$$

Given equation is Cauchy-Euler equation.

$$x^3 D^3 = \theta(\theta - 1)(\theta - 2)$$

$$x^2 D^2 = \theta(\theta - 1) \text{ where } \theta = \frac{d}{dz} \text{ and } z = \ln(x)$$

$$[\theta(\theta - 1)(\theta - 2) + 2\theta(\theta - 1) + 2]y = 0$$

$$(\theta^3 - \theta^2 + 2\theta + 2)y = 0$$

AE is

$$m^3 - m^2 + 2 = 0$$

$$(m+1)(m^2 - 2m + 2) = 0$$

$$m = -1, m = 1 \pm i$$

$$\text{C.F.} = C_1 e^{-z} + e^z (C_2 \cos z + C_3 \sin z)$$

Solution is

$$y = C_1 x^{-1} + x(C_2 \cos(\ln x) + C_3 \sin(\ln x)).$$



5

Complex Variables



Detailed Explanation of Try Yourself Questions

T1 : Solution

(b)

$$\begin{aligned}\frac{(2-3i)}{(-5+i)} &= \frac{(2-3i)}{(-5+i)} \times \frac{(-5-i)}{(-5-i)} \\ &= \frac{-10-2i+15i-3}{25+1} = \frac{-13+13i}{26} \\ &= -0.5 + 0.5i\end{aligned}$$

T2 : Solution

(a)

Given : $\alpha \neq 1$ and $\alpha^5 = 1$

$\therefore \alpha$ is fifth root of unity.

Hence, $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$

T3 : Solution

(b)

$$\begin{aligned}\frac{(2+i)^2}{(3-i)^2} &= \frac{4-1+4i}{9-1-6i} = \frac{3+4i}{8-6i} \\ &= \frac{3+4i}{2(4-3i)} \times \frac{4+3i}{4+3i} \\ &= \frac{12-12+25i}{2(16+9)} = \frac{i}{2} = \frac{1}{2}[e^{i\pi/2}]\end{aligned}$$

T4 : Solution

(c)

$$\begin{aligned}
 & 2^n \left[\frac{\sqrt{3}}{2} + \frac{i}{2} \right]^n + 2^n \left[\frac{\sqrt{3}}{2} - \frac{i}{2} \right]^n \\
 &= 2^n \left[\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right]^n + 2^n \left[\cos \frac{\pi}{6} - i \sin \frac{\pi}{6} \right]^n \\
 &= 2^n e^{in\pi/6} + 2^n e^{-in\pi/6} \\
 &= 2^n \left[2 \cos \frac{n\pi}{6} \right] = 2^{n+1} \cos \frac{n\pi}{6}
 \end{aligned}$$

T5 : Solution

$$f(z_1) = \frac{az_1 + b}{cz_1 + d} \text{ and } f(z_2) = \frac{az_2 + b}{cz_2 + d}$$

$$\frac{az_1 + b}{cz_1 + d} = \frac{az_2 + b}{cz_2 + d}$$

$$acz_1z_2 + bcz_2 + adz_1 + bd = acz_1z_2 + bcz_1 + adz_2 + bd$$

$$bc(z_2 - z_1) = ad(z_2 - z_1)$$

$$z_2 \neq z_1$$

$$\Rightarrow bc = ad$$

$$d = \frac{bc}{a} = \frac{4 \times 5}{2} = 10$$

T6 : Solution

(d)

$$f = u + iv$$

$$u = 3x^2 - 3y^2$$

for f to be analysis, we have Cauchy-Riemann conditions,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \dots(i)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \dots(ii)$$

From (i) we have

$$6x = \frac{\partial v}{\partial y}$$

$$\Rightarrow \int \partial v = \int 6x \partial y$$

$$v = 6xy + f(x)$$

$$\text{i.e. } v = 6xy + f(x)$$

 $\dots(iii)$

Now applying equation (ii) we get

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{-\partial v}{\partial x} \\ \Rightarrow -6y &= -\left[6x + \frac{df}{dx}\right] \\ \Rightarrow 6x + \frac{df}{dx} &= 6y \\ \frac{df}{dx} &= 6y - 6x\end{aligned}$$

By integrating,

$$f(x) = 6yx - 3x^2 + K$$

Substitute in equation (iii)

$$\begin{aligned}v &= 3x^2 + 6yx - 3x^2 + K \\ \Rightarrow v &= 6yx + K\end{aligned}$$

T7 : Solution

(c)

As per Cauchy-Riemann equation

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \text{and } \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial x} &= 2y \\ \text{and } \frac{\partial u}{\partial y} &= 2x \\ \frac{\partial v}{\partial y} &= 2y \\ \Rightarrow v &= y^2 + f(x) \\ \frac{\partial v}{\partial x} &= 0 + f'(x) = -2x \\ \therefore f(x) &= -x^2 + \text{constant} \\ \therefore v &= y^2 - x^2 + \text{constant}\end{aligned}$$

T8 : Solution

$$\begin{aligned}f(z) &= \frac{\sin(z-1)}{z-1} \\ \text{Maclaurin expansion of } \sin(z-1) &= (z-1) - \frac{(z-1)^3}{3!} + \frac{(z-1)^5}{5!} \dots \\ f(z) &= \left[1 - \frac{(z-1)^2}{3!} + \frac{(z-1)^4}{5!} \dots \right]\end{aligned}$$

\therefore There is no terms in the principal root of $f(z)$. Hence function has removable singularity.

T9 : Solution

(c)

$$x(z) = \frac{1-2z}{z(z-1)(z-2)}$$

poles are
 $z = 0, z = 1$ and $z = 2$

Residue at $z = 0$

$$\begin{aligned} \text{residue} &= \text{value of } \frac{1-2z}{(z-1)(z-2)} \text{ at } z = 0 \\ &= \frac{1-2 \times 0}{(0-1)(0-2)} = \frac{1}{2} \end{aligned}$$

Residue at $z = 1$

$$\begin{aligned} \text{residue} &= \text{value of } \frac{1-2z}{z(z-2)} \text{ at } z = 1 \\ &= \frac{1-2 \times 1}{1(1-2)} = 1 \end{aligned}$$

Residue at $z = 2$

$$\begin{aligned} \text{residue} &= \text{value of } \frac{1-2z}{z(z-1)} \text{ at } z = 2 \\ &= \frac{1-2 \times 2}{2(2-1)} = -\frac{3}{2} \end{aligned}$$

∴ The residues at its poles are $\frac{1}{2}, 1$ and $-\frac{3}{2}$.

T10 : Solution

(c)

$$f(z) = \frac{2z}{(z-1)^2(z-2)}$$

$z = 1$ is pole of order 2
 $z = 2$ pole of order 1

$$\begin{aligned} \text{Res}_{z=1} f(z) &= \lim_{z \rightarrow 1} \frac{1}{1!} \frac{d}{dz} \left((z-1)^2 \cdot \frac{2z}{(z-1)^2(z-2)} \right) \\ &= \lim_{z \rightarrow 1} \left(\frac{(z-2)(2) - 2z(1)}{(z-2)^2} \right) \\ &= \frac{-2-2}{(1-2)^2} = -4 \end{aligned}$$

$$\begin{aligned} \text{Res}_{z=2} f(z) &= \lim_{z \rightarrow 2} (z-2) \cdot \frac{2z}{(z-1)^2(z-2)} \\ &= \frac{4}{(2-1)^2} = 4 \end{aligned}$$

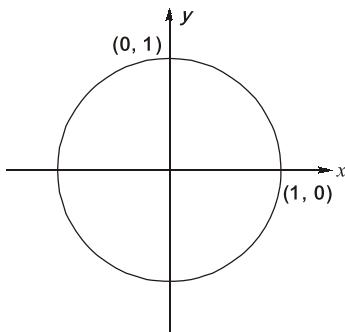
Sum of residues

$$= -4 + 4 = 0$$

T11 : Solution

(b)

$$\int_C f(z) dz = 2\pi i \text{ (sum of residues of } f(z) \text{ at all its pole lies inside } C)$$



$z = 0$ is the only pole inside C .

$$\begin{aligned} \operatorname{Res}_{z=0} f(z) &= \lim_{z \rightarrow 0} (z-0) \frac{\cos z}{z(z-2)(z-4)} \\ &= \frac{1}{(-2)(-4)} = \frac{1}{8} \end{aligned}$$

$$\begin{aligned} \int_{|z|=1} \frac{\cos z dz}{z(z-2)(z-4)} &= 2\pi i \times \frac{1}{8} \\ &= \frac{\pi i}{4} \end{aligned}$$

T12 : Solution

(a)

$$I = \int_C \frac{e^{2z}}{(z+1)^4} dz \text{ where } C \Rightarrow |z| = 3$$

$z = -1$ is the point inside C .

$$\begin{aligned} \therefore \int \frac{f(z)}{(z-a)^{n+1}} dz &= \frac{2\pi i}{n!} f^n(a) \\ f(z) &= e^{2z} \\ f'''(z) &= 8e^{2z} \end{aligned}$$

$$f'''(-1) = 8e^{-2}$$

$$\begin{aligned} \therefore \int_C \frac{e^{2z}}{(z+1)^4} dz &= \frac{2\pi i}{3!} \times 8e^{-2} \\ &= \frac{16\pi i}{6} \times e^{-2} \\ &= \frac{8\pi i}{3} \times e^{-2} \end{aligned}$$

T13 : Solution

(b)

(i)

 $Z_0 = 2$ lies inside C ,

So

$$\begin{aligned} \text{Res } f(z) &= \lim_{z \rightarrow 2} (z-2) \cdot \frac{e^z}{z-2} \\ &= e^2 = 7.39 \end{aligned}$$

$$\frac{1}{2\pi i} \int_C \frac{e^z}{z-2} dz = 2\pi i \cdot \frac{1}{2\pi i} (7.39) = 7.39$$

(ii)

 $Z_0 = -2$ lies outside C then
Res $f(z) = 0$

So

$$\int_C \frac{e^z}{z-2} dz = 2\pi i \frac{1}{2\pi i} (0) = 0$$

T14 : Solution

(b)

Singularities,

$$z = \frac{1}{2}, 2 \pm i$$

only,

$$z = \frac{1}{2} \text{ lies inside } C$$

By residue theorem,

$$\oint_C = 2\pi i(R) = \frac{48\pi i}{13}$$

Residue at $\frac{1}{2} = R_{1/2}$

$$= \lim_{z \rightarrow 1/2} \left[\left(z - \frac{1}{2} \right) \cdot \frac{2z+5}{\left(z - \frac{1}{2} \right) (z^2 + 4z + 5)} \right] = \frac{24}{13}$$

$$\text{Then the value of integral} = 2\pi i \times \frac{24}{13} = \frac{48\pi i}{13}$$

T15 : Solution

(b)

From the diagram C is $y = x$

$$\begin{aligned} I &= \int_C (x^2 + iy^2) dz \\ &= \int_C (x^2 + iy^2)(dx + idy) \\ &= \int_C (x^2 + ix^2)(dx + idx) \\ &= \int x^2 dx + ix^2 dx + ix^2 dx - x^2 dx \\ &= 2i \int_0^1 x^2 dx = 2i \left(\frac{x^3}{3} \right)_0^1 = \frac{2i}{3} \end{aligned}$$

T16 : Solution

$$\begin{aligned} I &= \oint_C \frac{-3z+4}{(z^2+4z+5)} dz \\ &= 2\pi i (\text{sum of residues}) \end{aligned}$$

Poles of $\frac{-3z+4}{(z^2+4z+5)}$ are given by

$$\begin{aligned} z^2 + 4z + 5 &= 0 \\ z &= \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2} \\ &= -2 \pm i \end{aligned}$$

Since the poles lie outside the circle $|z| = 1$.

So $f(z)$ is analytic inside the circle $|z| = 1$.

Hence $\oint_C f(z) dz = 2\pi i (0) = 0$

T17 : Solution

(a)

$$\text{Given : } \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 2x + 2} dx$$

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \frac{\sin z}{z^2 + 2z + 2} dz \\ \sin z &= \text{imaginary part of } e^{iz} \\ &= \int_{-\infty}^{\infty} \frac{\text{I.P of } e^{iz}}{z^2 + 2z + 2} dz \end{aligned}$$

Poles are $z^2 + 2z + 2 = 0$

$$z = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$z = -1 - i$$



Outside upper half



Residue is 0

$$-1 + i$$



inside upperhalf

Res $\phi(z)$

$$z = -1 + i$$

$$= \lim_{z \rightarrow -1+i} z - (-1+i) \frac{e^{iz}}{(z - (-1+i))(z - (-1-i))}$$

$$= \frac{e^{i(-1+i)}}{(-1+i) - (-1-i)} = \frac{e^{-i-1}}{-1+i+1+i} = \frac{e^{-i-1}}{2i}$$

$$I = \text{I.P. of } 2\pi i \left(\frac{e^{-i-1}}{2i} \right) = \text{I.P. of } \pi (e^{-i} \cdot e^{-1})$$

$$= \text{I.P. of } \pi e^{-1} (\cos 1 - i \sin 1) = \frac{-\pi \sin 1}{e}$$



6

Probability and Statistics



Detailed Explanation of Try Yourself Questions

T1 : Solution

$$D = \{x_1, x_2, x_3, x_4\}$$

Total number of vector possible

$$= 2 \times 2 \times 2 \times 2 = 16$$

Combinations that product of first and third coordinate is '0' is

$$\begin{array}{|c|c|c|c|} \hline 0 & x_2 & 1 & x_3 \\ \hline \end{array} = 2 \times 2$$
$$\begin{array}{|c|c|c|c|} \hline 1 & x_2 & 0 & x_3 \\ \hline \end{array} = 2 \times 2$$
$$\begin{array}{|c|c|c|c|} \hline 0 & x_2 & 0 & x_3 \\ \hline \end{array} = 2 \times 2$$

Probability that product of first and third coordinate is '0' is

$$= \frac{4+4+4}{16} = \frac{12}{16} = \frac{3}{4}$$

T2 : Solution

Let $P(A)$ is probability of getting plumbing contract and $P(B)$ is probability of getting electrical contract.

$$P(A) = 0.5 \quad P(B) = 0.3$$

$$\overline{P(A \cup B)} = 0.25$$

$$P(A \cup B) = 1 - \overline{P(A \cup B)}$$

$$= 1 - 0.25 = 0.75$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = 0.5 + 0.3 - 0.75 = 0.05$$

T3 : Solution

	X	Y
Supply	60%	40%
Probability of supply	$\frac{60}{100} = \frac{3}{5}$	$\frac{40}{100} = \frac{2}{5}$
Reliable	96%	72%
Reliability Probability	$\frac{96}{100}$	$\frac{72}{100}$

By Baye's theorem,

Probability that shock absorber is reliable and from by

$$Y = \frac{\frac{72}{100} \times \frac{2}{5}}{\frac{72}{100} \times \frac{2}{5} + \frac{96}{100} \times \frac{3}{5}} = \frac{1}{3}$$

$$= 0.0334$$

T4 : Solution

$\boxed{4B \ 3W}$ $\boxed{B \ W \ B \ W \ B \ W \ B}$ is the only one arrangement is favourable arrangement out of

$\frac{7!}{3! 4!}$ arrangements.

$$\therefore \text{Required prob.} = \frac{1}{\frac{7 \times 6 \times 5 \times 4!}{3! 4!}} = \frac{1}{35}$$

T5 : Solution

$\boxed{5 \text{ Defective} \quad 15 \text{ Non-defective}}$

$$\text{Required probability} = \frac{\frac{15}{20} C_1 \cdot \frac{14}{19} C_1}{C_1 \cdot \frac{19}{20} C_1} = \frac{210}{380} = \frac{21}{38}$$

T6 : Solution

$$\begin{aligned} \Sigma P(x) &= 1 \\ 10K^2 + 9K &= 1 \\ 10K^2 + 9K - 1 &= 0 \\ 10K^2 + 10K - K - 1 &= 0 \\ 10K(K+1) - (K+1) &= 0 \\ (K+1)(10K-1) &= 0 \\ K &= -1 \quad K = \frac{1}{10} \end{aligned}$$

Probability any event is non-negative.

$$\therefore K = 0.10$$

T7 : Solution

Sum of square of first 23 natural numbers

$$= \frac{(n)(n+1)(2n+1)}{6} \Big|_{n=23}$$

Mean of square of first 23 natural numbers

$$= \frac{\text{Sum}}{23} = \frac{(23)(23+1)(46+1)}{6 \times 23}$$

$$= 188$$

T8 : Solution

(c)

$$\text{Probability of appearing head} = \frac{1}{2}$$

$$\text{Probability of appearing tail} = \frac{1}{2}$$

Out of 8, probability of getting atleast 5 heads

$$= {}^8C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^3 + {}^8C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^2$$

$$+ {}^8C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^1 + {}^8C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^0$$

$$= ({}^8C_5 + {}^8C_6 + {}^8C_7 + {}^8C_8) \left(\frac{1}{2}\right)^8 = \frac{93}{256}$$

T9 : Solution

$$n = 5$$

$$P = P(H) = \frac{1}{2}$$

$$q = 1 - \frac{1}{2} = \frac{1}{2}$$

Probability of getting exactly two heads.

$$P(x=2) = {}^nC_2 P^2 q^{n-2}$$

$$= {}^5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3$$

$$= \frac{10}{32} = \frac{5}{16}$$

T10 : Solution

(a)

Given

$$f(z) = \frac{2z}{(z-1)(z-2)}$$

Given that,

$$P(x=1) = \frac{2}{3}P(x=2)$$

∴ For Poisson distribution $P(X=x) = \left[\frac{e^{-\lambda} \lambda^x}{x!} \right]$; where λ is mean value of the distribution

$$\therefore \left[\frac{e^{-\lambda} \lambda^1}{1!} \right] = \frac{2}{3} \left[\frac{e^{-\lambda} \lambda^2}{2!} \right]$$

$$\Rightarrow \frac{\lambda}{1} = \frac{2 \lambda^2}{3 \cdot 2}$$

$$\Rightarrow \lambda = 3$$

For Poisson distribution Mean = Variance

Hence, Variance = 3

T11 : Solution

(b)

$$n = 100, P = 0.01, \lambda = \text{Mean} = nP = 1$$

$$\text{Required probability} = 1 - [P(x=0) + P(x=1) + P(x=2)]$$

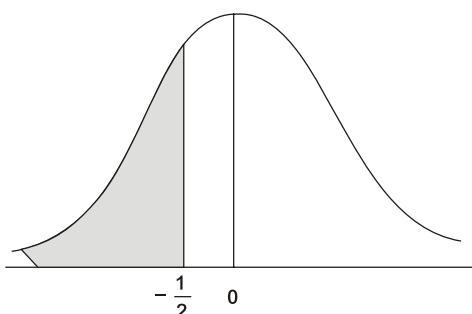
$$\begin{aligned} &= 1 - \left[\frac{e^{-1}(1)^0}{0!} + \frac{e^{-1}(1)^1}{1!} + \frac{e^{-1}(1)^2}{2!} \right] \\ &= 1 - \left[e^{-1} + e^{-1} + \frac{e^{-1}}{2} \right] \\ &= 1 - \frac{5e^{-1}}{2} \end{aligned}$$

T12 : Solution

(b)

$$\text{Here, } \sigma^2 = 4 \Rightarrow \sigma = 2$$

$$P(x < 0) = P\left(z < \frac{0-\mu}{\sigma}\right) = P\left(z < \frac{0-1}{2}\right) = P\left(z < -\frac{1}{2}\right)$$



Which is the shaded area in the picture and its value is clearly between 0. and 0.5.

T13 : Solution

(a)

$$\text{Given: } f(x) = \frac{1}{5}e^{-x/5}, x \geq 0$$

Comparing the general function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}; & x \geq 0 \\ 0; & x < 0 \end{cases} \Rightarrow \lambda = \frac{1}{5}$$

The cumulative distributive function

$$\begin{aligned} f(a) &= 1 - e^{-\lambda a} \\ \Rightarrow P(E > 5) &= 1 - f(5) \\ &= 1 - (1 - e^{-\lambda \cdot 5}) \\ &= e^{-\frac{1}{5} \times 5} = e^{-1} = 1/e \end{aligned}$$

Alternate solution:

$$\begin{aligned} P(x > 5) &= \int_5^{\infty} f(x) dx = \int_5^{\infty} \frac{1}{5} e^{-x/5} dx \\ &= \frac{1}{5} \int_5^{\infty} e^{-x/5} dx \\ &= \frac{1}{5} \left[-5e^{-x/5} \Big|_5^{\infty} \right] = -1 \left[e^{-\infty} - e^{-5/5} \right] \\ &= e^{-1} = \frac{1}{e} \end{aligned}$$

T14 : Solution

(a)

$$\text{Mean}(x_1) = 0.5$$

$$\frac{1}{\lambda_1} = 0.5$$

$$\lambda_1 = \frac{1}{0.5} = 2$$

$$\text{Mean}(x_2) = 0.25$$

$$\frac{1}{\lambda_2} = 0.25$$

$$\lambda_2 = \frac{1}{0.25} = 4$$

$$y = \text{mean}(x_1, x_2)$$

$$\text{Mean}(y) = \frac{1}{\lambda_1 + \lambda_2} = \frac{1}{2 + 4} = \frac{1}{6}$$



7

Numerical Methods



Detailed Explanation of Try Yourself Questions

T1 : Solution

Here,

$$\begin{aligned}x_0 &= 2 \\f(x) &= x^3 - x^2 + 4x - 4 \\f'(x) &= 3x^2 - 2x + 4 \\f(x_0) &= f(2) = 8 \\f'(x_0) &= f'(2) = 12\end{aligned}$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{8}{12} = \frac{4}{3}$$

T2 : Solution

$$\begin{aligned}x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\x = 2, \quad f(x_0) &= 2 + \sqrt{2} - 3 = \sqrt{2} - 1\end{aligned}$$

$$f'(x) = 1 + \frac{1}{2\sqrt{x}}$$

$$f'(x_0) = 1 + \frac{1}{2\sqrt{2}}$$

$$\begin{aligned}\text{Then, } x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{\sqrt{2} - 1}{1 + \frac{1}{2\sqrt{2}}} \\&\Rightarrow x_1 = 1.694\end{aligned}$$

T3 : Solution

The equation is $f(x) = x^2 - 13 = 0$
Newton-Raphson iteration equation is

$$\begin{aligned}
 x_1 &= x_0 - \left[\frac{f(x_0)}{f'(x_0)} \right] \\
 f(x_0) &= x_0^2 - 13 \\
 f'(x_0) &= 2x_0 \\
 \therefore & \quad x_1 = x_0 - \left[\frac{x_0^2 - 13}{2x_0} \right] = \frac{x_0^2 + 13}{2x_0} \\
 \text{put } & \quad x_0 = 3.5 \text{ (as given)} \\
 & \quad x_1 = \frac{3.5^2 + 13}{2 \times 3.5} = 3.607 \\
 \therefore & \quad \text{The approximation after one iteration} = 3.607
 \end{aligned}$$

T5 : Solution

(b)

$$\begin{aligned}
 u(x_1, x_2) &= 10x_2 \sin x_1 - 0.8 = 0 \\
 v(x_1, x_2) &= 10x_2^2 - 10x_2 \cos x_1 - 0.6 = 0
 \end{aligned}$$

The Jacobian matrix is

$$\begin{aligned}
 \begin{bmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} \end{bmatrix} &= \begin{bmatrix} 10x_2 \cos x_1 & 10 \sin x_1 \\ 10x_2 \sin x_1 & 20x_2 - 10 \cos x_1 \end{bmatrix} \\
 &= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}
 \end{aligned}$$



8

Laplace Transforms



Detailed Explanation of Try Yourself Questions

T1 : Solution

(b)

$$\begin{aligned} & L(\sin^2 t) \\ & L\left(\frac{1-\cos 2t}{2}\right) \\ & L\left(\frac{1}{2}\right) - L\left(\frac{\cos 2t}{2}\right) \\ & \frac{1}{2}L(1) - \frac{1}{2}L(\cos 2t) \\ & \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{s}{s^2 + 4} = \frac{s^2 + 4 - s^2}{2s(s^2 + 4)} \\ & = \frac{4}{2s(s^2 + 4)} = \frac{2}{s(s^2 + 4)} \end{aligned}$$

T2 : Solution

(b)

$$\begin{aligned} L[U(t-a)] &= \int_0^\infty e^{-st} U(t-a) dt \\ &= \int_0^a e^{-st} \cdot 0 \cdot dt + \int_a^\infty e^{-st} \cdot 1 \cdot dt \\ &= 0 + \int_a^\infty e^{-st} dt = \left[\frac{e^{-st}}{-s} \right]_a^\infty = \frac{e^{-as}}{s} \end{aligned}$$

T3 : Solution

(c)

By property of Laplace

$$\mathcal{L}(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}} = \frac{n!}{(s-a)^{n+1}}$$

$\therefore a = -2, n = 3$

So the laplace transform will be

$$= \frac{3!}{(s+2)^4} = \frac{6}{(s+2)^4}$$

T4 : Solution

(d)

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

Given that,

$$F(s) = \left[\frac{3s+1}{s^3 + 4s^2 + (K-3)s} \right]$$

$$\lim_{t \rightarrow \infty} f(t) = 1$$

$$\Rightarrow \lim_{s \rightarrow 0} s \left[\frac{3s+1}{s^3 + 4s^2 + (K-3)s} \right] = 1$$

$$\Rightarrow \lim_{s \rightarrow 0} \left[\frac{3s+1}{s^2 + 4s + (K-3)} \right] = 1$$

$$\Rightarrow \frac{1}{K-3} = 1$$

$$\Rightarrow K-3 = 1$$

$$\Rightarrow K = 4$$

T5 : Solution

(c)

By property of Laplace

$$\therefore \mathcal{L}(\cos at) = \frac{s}{s^2 + a^2}$$

$$\therefore \mathcal{L}^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at$$

In the given question $a = 2$

$$\text{Hence, } \mathcal{L}^{-1}\left(\frac{s^2}{s^2 + 4}\right) = \cos 2t$$



10

Partial Differential Equations



Detailed Explanation of Try Yourself Questions

T1 : Solution

(a)

Given equation is

$$\frac{\partial u}{\partial x} = 4 \frac{\partial u}{\partial y} \quad \dots(1)$$

Let

where

$$u(x, y) = XY \quad \dots(2)$$

$$X = f(x) \text{ and } Y = g(y)$$

$$\frac{\partial u}{\partial x} = X'Y; \quad \frac{\partial u}{\partial y} = XY'$$

$$X'Y = 4XY$$

$$\frac{X'}{X} = 4, \quad \frac{Y'}{Y} = K$$

$$\frac{X'}{X} = K, \quad \frac{Y'}{Y} = \frac{K}{4}$$

$$X = C_1 e^{Kx}, \quad Y = C_2 e^{Ky/4}$$

∴

$$u(x, y) = XY$$

$$= C_1 C_2 e^{Kx} e^{Ky/4}$$

$$u(0, y) = C_1 C_2 e^{Ky/4} = 8e^{-3y}$$

Now,

$$C_1 C_2 = 8, \quad \frac{K}{4} = -3$$

$$K = -12$$

$$u(x, y) = 8e^{-12x} e^{-3y}$$

$$u(x, y) = 8e^{-12x-3y}$$

