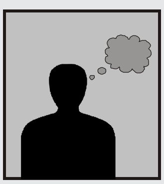
2020

RANK improvement WORKBOOK



Answer key and Hint of
Objective & Conventional *Questions*

Electrical Engineering

Control Systems

MADE EASY Publications

MADE EASY
Publications

1 Basics of Control Systems

LEVEL 1 Objective Solutions

1.	(1)
2.	(a)
3.	(d)
4.	(c)
5.	(b)
6.	(d)
7.	(b)
8.	(b)
9.	(a)
10.	(a)

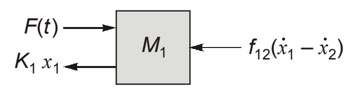
LEVEL 2 Objective Solutions

11.	(c)
12.	(a)
13.	(c)
14.	(d)
15.	(c)
16.	(a)
17.	(a)
18.	(100)
19.	(9×10^{-3})
20.	(b)

LEVEL 3 Conventional Solutions

Solution: 1

Differential equation for mass M_1:

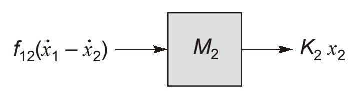


Free Body Diagram of M_1

$$F(t) - k_1x_1 - f_{12}(\dot{x}_1 - \dot{x}_2) = M_1\ddot{x}_1$$

$$\Rightarrow \quad M_1\ddot{x}_1 + f_{12}(\dot{x}_1 - \dot{x}_2) + K_1x_1 = F(t) \quad \text{...(i)}$$

Differential equation for M_2:



Free Body Diagram of M_2

$$f_{12}(\dot{x}_1 - \dot{x}_2) - K_2x_2 = M_2\ddot{x}_2$$

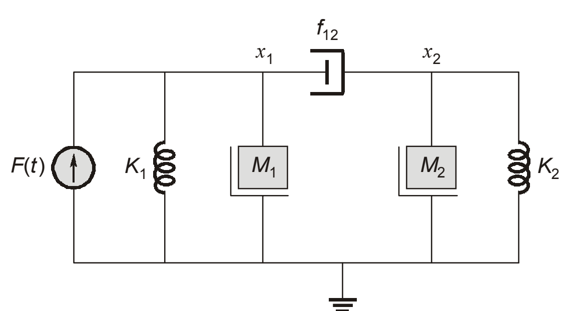
$$\Rightarrow \quad M_2\ddot{x}_2 - f_{12}(\dot{x}_1 - \dot{x}_2) + K_2x_2 = 0 \quad \text{...(ii)}$$

(i) Differential equations of the mechanical system are:

$$M_1\ddot{x}_1 + f_{12}(\dot{x}_1 - \dot{x}_2) + K_1x_1 = F(t)$$

and $M_2\ddot{x}_2 - f_{12}(\dot{x}_1 - \dot{x}_2) + K_2x_2 = 0$

(ii) Mechanical equivalent representation is shown below:



(iii) Standard force equation:

$$M\ddot{x} + f\dot{x} + Kx = F \quad \text{...(iii)}$$

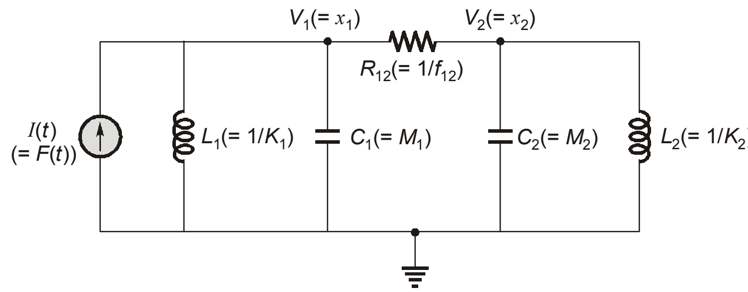
Standard current equation:

$$C\ddot{\phi} + \frac{1}{R}\dot{\phi} + \frac{\phi}{L} = I \quad \text{...(iv)}$$

Comparing the above equations (iii) and (iv) we get,

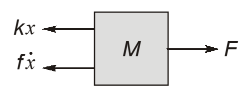
$M \to C, \quad f \to 1/R, \quad K \to 1/L, \quad F \to I$

So, the electrical analogous circuit based on the force-current analogy is given below,



Solution: 2

Consider the FBD of given system as:



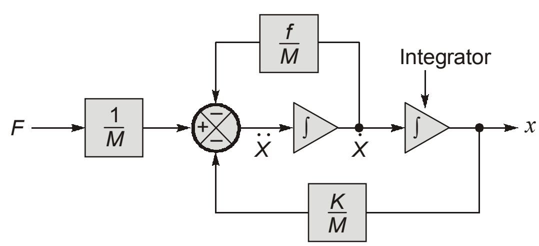
Free Body Diagram of M

Differential equation of the system:

$$\Rightarrow \quad F - Kx - f\dot{x} = M\ddot{x}$$

$$\Rightarrow \quad M\ddot{x} + f\dot{x} + Kx = F$$

Now, the integrator based electronic circuit to simulate the above mechanical system is shown below,



$$\Rightarrow \quad \boxed{\ddot{X} + \frac{f}{M}\dot{X} + \frac{K}{M}X = F \cdot \frac{1}{M}}$$

Solution : 3

For the given close loop system, $G(s) = \dfrac{10}{s(s+1)}$,

$$H(s) = 5$$

$$T(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{10}{s(s+1)}}{1+\frac{10}{s(s+1)}\times 5} = \frac{10}{s^2+s+50}$$

The sensitivity of $T(s)$ with respect to $G(s)$ is given by,

$$S_G^T = \frac{1}{1+G(s)H(s)} = \frac{s(s+1)}{s^2+s+50}$$

To calculate at S_G^T at $\omega = 1$, substitute $s = j\omega$ to convert the time domain function to frequency domain.

$$\therefore \quad S_G^T = \frac{j\omega(1+j\omega)}{(j\omega)^2+j\omega+50} = \frac{-\omega^2+j\omega}{(50-\omega^2)+j\omega}$$

For the value of $\omega = 1$

$$S_G^T = \frac{-1+j1}{49+j1}$$

$$\Rightarrow \qquad \left|S_G^T\right| = \frac{\sqrt{1+1}}{\sqrt{49^2+1}} = 0.02885$$

The sensitivity of $T(s)$, with respect to $H(s)$ is given by,

$$S_H^T = \frac{-G(s)H(s)}{1+G(s)H(s)} = \frac{-\dfrac{10}{s(s+1)}\times 5}{1+\dfrac{10}{s(s+1)}\times 5} = \frac{-50}{s^2+s+50}$$

Replacing 'S' by $j\omega$ to convert the function to the frequency domain, we get,

$$S_H^T = \frac{-50}{(j\omega)^2+j\omega+50} = \frac{-50}{(50-\omega^2)+j\omega}$$

At $\omega = 1$,

$$S_H^T = \frac{-50}{49+j1}$$

$$\left|S_H^T\right| = \frac{|-50|}{\sqrt{49^2+1}} = 1.0202$$

It can be observed that S_H^T is more than S_G^T i.e. output of the system is more sensitive to the variation in $H(s)$ rather than $G(s)$.

Solution : 4

The transfer functions has three poles and one zero, therefore the transfer function consists of one term in the numerator and three terms in the denominator.

The poles are located at $s = 0$, $s = -2$, $s = -4$ and zero is located at $s = -3$.

The transfer function is thus,

$$G(s) = \frac{K(s+3)}{s(s+2)(s+4)}$$

It is given that $s = 1$, the value of $G(s)$ is 3.2.

$$G(1) = 3.2 = \frac{K(1+3)}{s(1+2)(1+4)} = \frac{K\times 4}{1\times 3\times 5}$$

$$K = \frac{3.2\times 1\times 3\times 5}{4} = 12$$

$$\therefore \qquad G(s) = \frac{12(s+3)}{s(s+2)(s+4)}$$

Solution : 5

The network equation are,

$$e_i = Ri + \frac{1}{c}\int i\,dt \qquad \text{...(i)}$$

and

$$e_0 = \frac{1}{c}\int i\,dt \qquad \text{...(ii)}$$

Assuming initial condition as zero and taking Laplace transform on both sides of equation (i) and (ii), the following equations are obtained:

$$E_i(s) = RI(s) + \frac{1}{C}\frac{I(s)}{s} \quad \text{...(iii)}$$

or
$$E_0(s) = \frac{1}{C}\frac{I(s)}{s} \quad \text{...(iv)}$$

$$E_i(s) = \left[R + \frac{1}{Cs}\right] I(s) \quad \text{...(v)}$$

and
$$E_0(s) = \frac{1}{Cs} I(s) \quad \text{...(vi)}$$

From equation (v) and (vi), the transfer function is obtained below.

$$\frac{E_0(s)}{E_i(s)} = \frac{\frac{1}{Cs} I(s)}{\left[R + \frac{1}{Cs}\right] I(s)}$$

or
$$\frac{E_0(s)}{E_i(s)} = \frac{1}{(RCs + 1)}$$

■■■■

2 Block Diagram and Signal Flow Graph

LEVEL 1 Objective Solutions

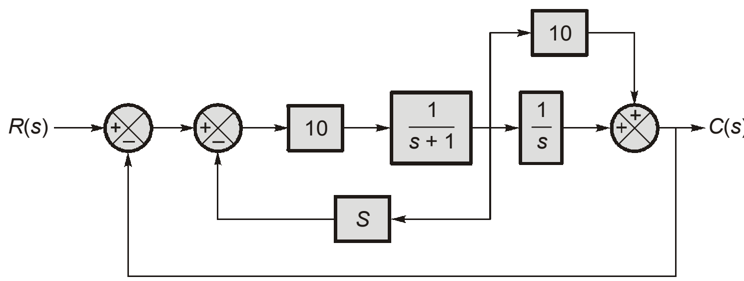
1. (d)
2. (c)
3. (a)
4. (c)
5. (c)
6. (a)
7. (a)
8. (a)
9. (b)
10. (d)
11. (d)
12. (c)
13. (b)
14. (0.117)

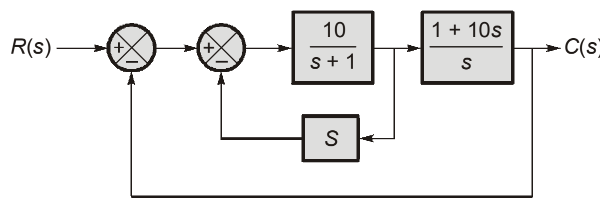
LEVEL 2 Objective Solutions

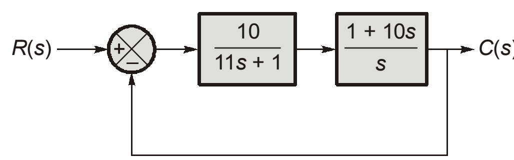
15. (d)
16. (b)
17. (–0.5)
18. (b)
19. (a)
20. (c)
21. (d)
22. (a)
23. (a)
24. (c)
25. (b)
26. (a)

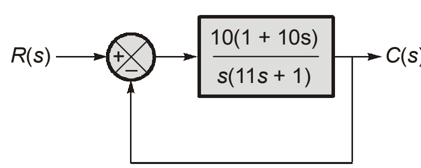
LEVEL 3 Conventional Solutions

Solution: 1







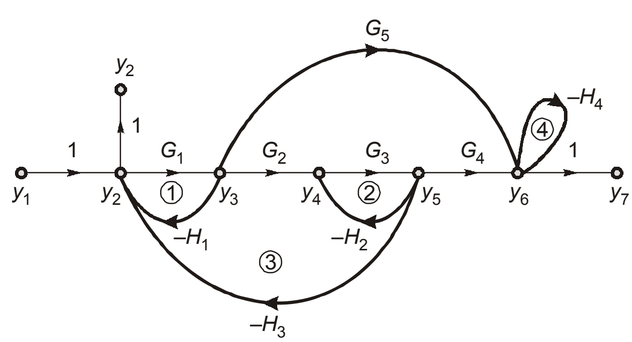


$$\therefore \qquad G(s) = \frac{10(10s+1)}{s(11s+1)}$$

$$H(s) = 1$$

Solution : 2

The given signal flow graph is as follows:



To determine $\frac{y_2}{y_1}$:

Using Mason's gain formula,

$$\frac{y_2}{y_1} = \frac{1}{\Delta}\sum_{i=1}^{N} P_i \Delta_i \;;\; N = \text{Number of forward paths}$$

Forward path gains, $P_1 = 1$

Individual loop gains,

$$L_1 = -G_1 H_1$$
$$L_2 = -G_3 H_2$$
$$L_3 = -G_1 G_2 G_3 H_3$$
$$L_4 = -H_4$$

Product of loop gain of two non touching loops,

$$L_1 L_2 = G_1 G_3 H_1 H_2$$
$$L_1 L_4 = G_1 H_1 H_4$$
$$L_2 L_4 = G_3 H_2 H_4$$
$$L_3 L_4 = G_1 G_2 G_3 H_3 H_4$$

There are no three non-touching loops.

So, $\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1 L_2 + L_1 L_4 + L_2 L_4 + L_3 L_4)$

$= 1 + G_1 H_1 + G_3 H_2 + G_1 G_2 G_3 H_3 + H_4 + G_1 G_3 H_1 H_2 + G_1 H_1 H_4 + G_3 H_2 H_4 + G_1 G_2 G_3 H_3 H_4$

$\Delta = (1 + H_4)(1 + G_1 H_1 + G_3 H_2 + G_1 G_2 G_3 H_3) + G_1 G_3 H_1 H_2$

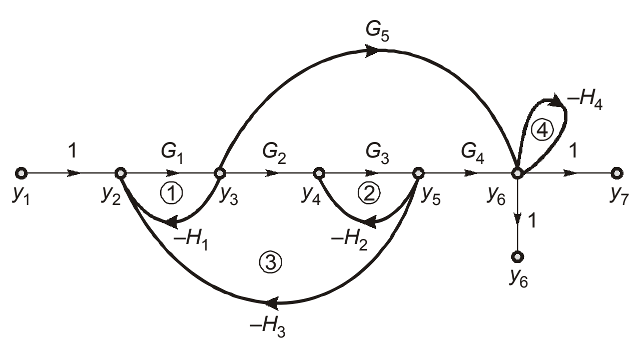
Loop-2 and loop-4 are non-touching to the forward path, hence,

$$\Delta_1 = 1 - (L_2 + L_4) + (L_2 L_4) = 1 + G_3 H_2 + H_4 + G_3 H_2 H_4 = (1 + H_4)(1 + G_3 H_2)$$

So, the gain $\frac{y_2}{y_1}$ can be given as,

$$\frac{y_2}{y_1} = \frac{P_1 \Delta_1}{\Delta} = \frac{(1+H_4)(1+G_3 H_2)}{G_1 G_3 H_1 H_2 + (1+H_4)(1+G_1 H_1 + G_3 H_2 + G_1 G_2 G_3 H_3)}$$

To determine $\frac{y_6}{y_1}$:



Forward path gains,

$$P_1 = G_1 G_2 G_3 G_4$$
$$P_2 = G_1 G_5$$

There is no non touching loops for forward path-1, so $\Delta_1 = 1$

Loop-2 is non touching to the forward path –2. So,

$$\Delta_2 = 1 - L_2 = 1 + G_3 H_2$$

The gain $\frac{y_6}{y_1}$ can be given as,

$$\frac{y_6}{y_1} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$= \frac{G_1G_2G_3G_4 + G_1G_5(1+G_3H_2)}{G_1G_3H_1H_2 + (1+H_4)(1+G_1H_1 + G_3H_2 + G_1G_2G_3H_3)}$$

To determine $\frac{y_7}{y_2}$:

$$\frac{y_7}{y_2} = \frac{y_7}{y_1}\times\frac{y_1}{y_2}$$

From the given signal flow graph, it is clear that,

$$\frac{y_7}{y_1} = \frac{y_6}{y_1}$$

So, the gain $\frac{y_7}{y_2}$ can be given as,

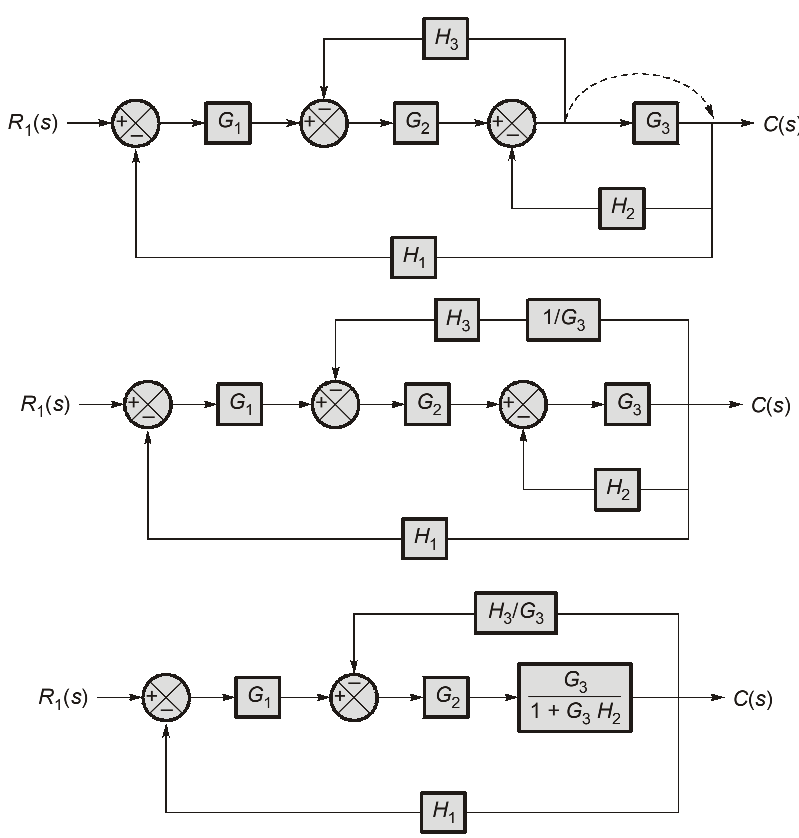
$$\frac{y_7}{y_2} = \frac{y_6}{y_1}\times\frac{y_1}{y_2}$$

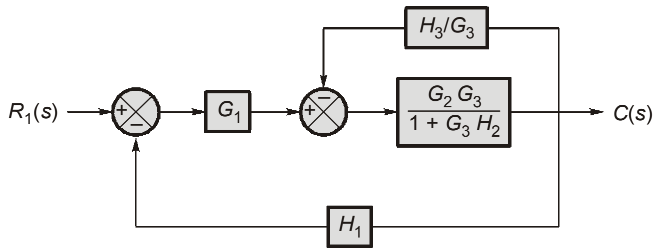
$$= \frac{G_1G_2G_3G_4 + G_1G_5(1+G_3H_2)}{\Delta}\times\frac{\Delta}{(1+H_4)(1+G_3H_2)}$$

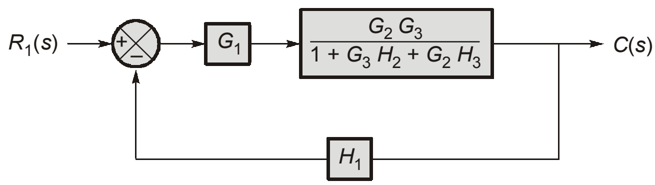
$$= \frac{G_1G_2G_3G_4 + G_1G_5(1+G_3H_2)}{(1+H_4)(1+G_3H_2)}$$

Solution : 3

When $R_2 = 0$





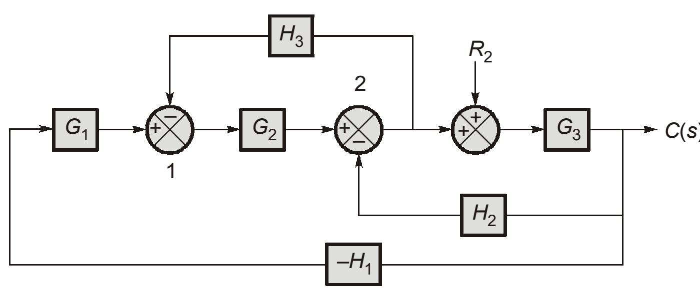


$$\frac{C(s)}{R_1(s)} = \frac{\dfrac{G_1G_2G_3}{1+G_3H_2+G_2H_3}}{1+\dfrac{G_1G_2G_3H_1}{1+G_3H_2+H_3G_2}}$$

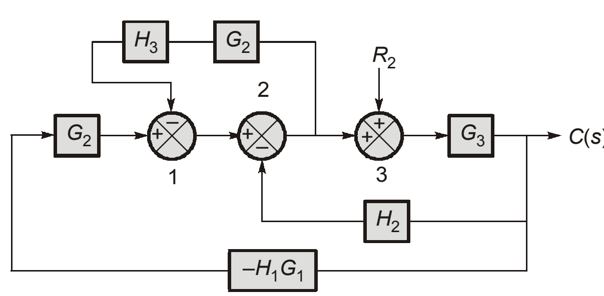
$$= \frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1}$$

(b) When $R_1 = 0$

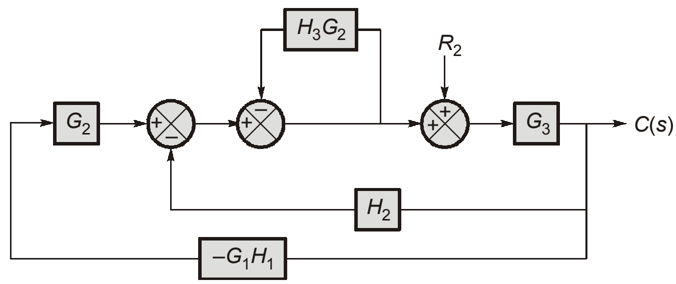
Now remove the summing point since feedback is negative therefore put sign with H_1.

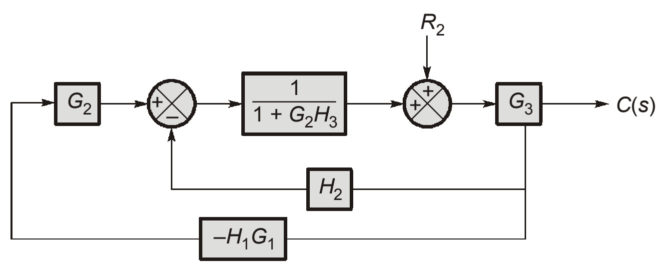


Shift the summering point 1 after G_2 then,

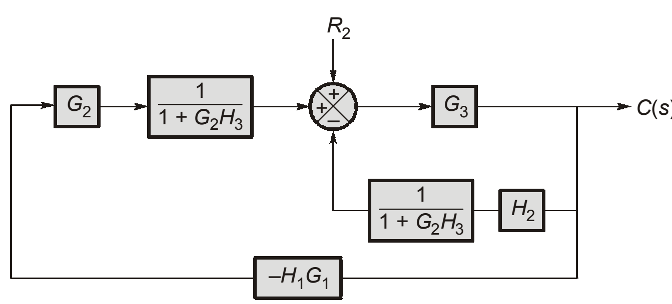


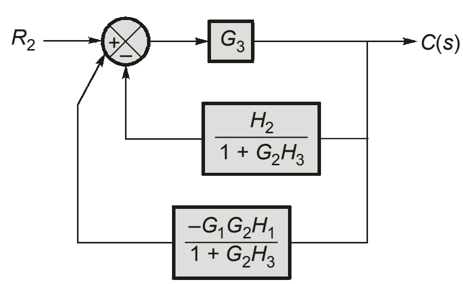
Now, interchange the summering point 1 and 2

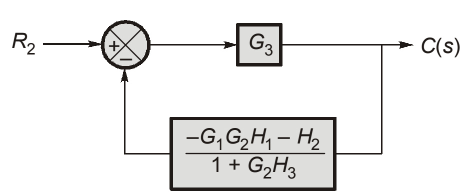




Shift the summering point after the block







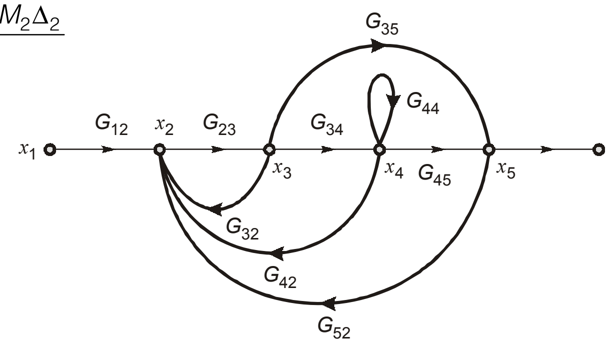
$$\frac{C(s)}{R_2(s)} = \frac{G_3}{1-\left[\frac{-G_1G_2H_1-H_2}{1+G_2H_3}\right]G_3} = \frac{G_3(1+G_2H_3)}{1+G_1G_2G_3H_1+G_3H_2+G_2H_3}$$

Solution : 4

For the given equations the SFG is drawn below.

Number of forward paths = 2

$$\frac{x_5}{x_1} = \frac{\sum_{K=1}^{2} M_k \Delta_k}{\Delta} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta}$$



The gain of the paths

$$M_1 = G_{12}G_{23}G_{34}G_{45}$$
$$M_2 = G_{12}G_{23}G_{35}$$

Gain of individual loops

$$L_1 = G_{23}G_{32}$$
$$L_2 = G_{23}G_{34}G_{42}$$
$$L_3 = G_{23}G_{34}G_{45}G_{52}$$
$$L_4 = G_{23}G_{35}G_{52}$$
$$L_5 = G_{44}$$

Gain of two non-touching loops

$$L_1L_5 = G_{23}G_{34}G_{44}$$
$$L_4L_5 = G_{23}G_{35}G_{52}G_{44}$$

Since, all the loops touch the forward path (i) $\therefore\ \Delta_1 = 1 - 0 = 1$

Loop L_5 do not touch the second forward path $\Delta_2 = 1 - G_{44}$

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5) + (L_1L_5 + L_4L_5)$$

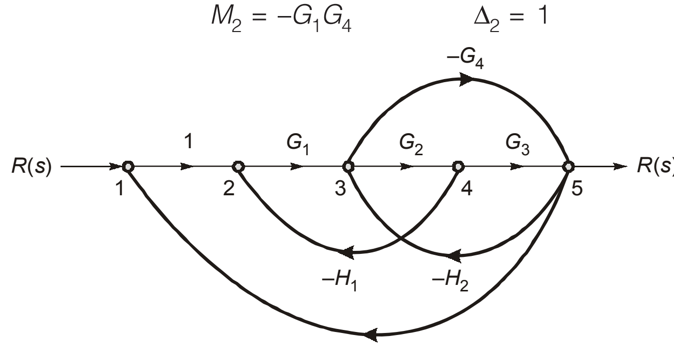
$$\therefore \quad \frac{x_5}{x_1} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta}$$

$$\frac{x_5}{x_1} = \frac{G_{12}G_{23}G_{34}G_{45} + G_{12}G_{23}G_{35}(1-G_{44})}{1-(G_{23}G_{35} + G_{23}G_{34}G_{42} + G_{23}G_{34}G_{45}G_{52} + G_{23} + G_{35} + G_{44}) + (G_{23}G_{32}G_{44} + G_{23}G_{35} - G_{52}G_{44})}$$

Solution : 5

For SFG is shown in figure. The gain of the forward paths.

$$M_1 = G_1G_2G_3 \qquad \Delta_1 = 1$$
$$M_2 = -G_1G_4 \qquad \Delta_2 = 1$$



Individual loops

$$L_1 = -G_1G_2H_1$$

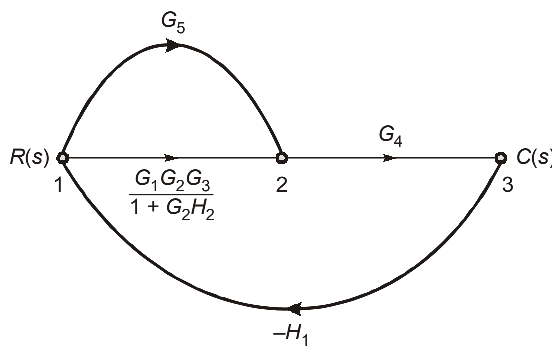
$$L_2 = -G_2G_3H_2$$
$$L_3 = -G_1G_2G_3$$
$$L_4 = G_1G_4$$
$$L_5 = G_4H_2$$

$$\frac{C}{R} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{G_1G_2G_3 - G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3 - G_1G_4 - G_4H_2}$$

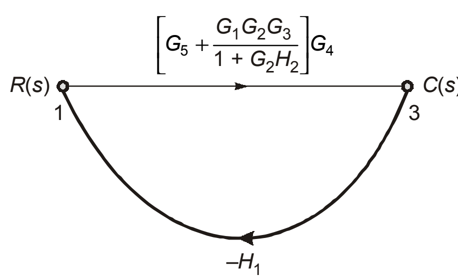
Solution : 6

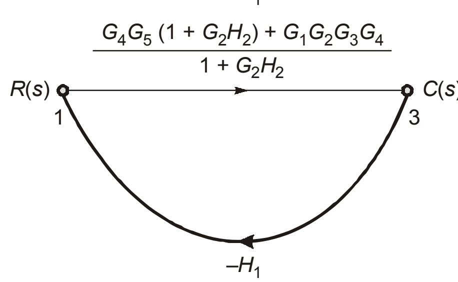
We will eliminate the nodes between $R(s)$ and $C(s)$ till one branch is left.

We get the following signal flow graph by reducing the inner feedback loop.



On further reduction, we get the following signal flow graph.





Hence,

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4 + G_4G_5(1+G_2H_2)}{1+G_2H_2 + (G_4G_5H_1)(1+G_2H_2) + G_1G_2G_3G_4H_1}$$

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3 Time Response Analysis

LEVEL 1 Objective Solutions

1.	(0.47)
2.	(a)
3.	(38.16)
4.	(1)
5.	(d)
6.	(a)
7.	(b)
8.	(a)
9.	(b)
10.	(b)
11.	(c)
12.	(b)
13.	(c)
14.	(b)
15.	(b)
16.	(c)
17.	(b)
18.	(d)

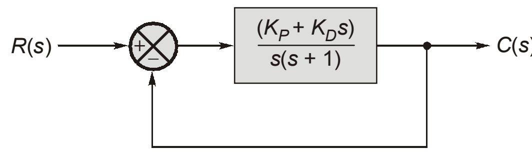
LEVEL 2 Objective Solutions

19.	(b)
20.	(b)
21.	(13.53)
22.	(12)
23.	(2.8)
24.	(d)
25.	(c)
26.	(c)
27.	(a)
28.	(d)
29.	(d)
30.	(a)
31.	(c)
32.	(d)
33.	(c)
34.	(b)
35.	(c)
36.	(a)

LEVEL 3 Conventional Solutions

Solution: 1

For $K_I = 0$



$$\frac{C(s)}{R(s)} = \frac{K_P + K_D s}{s(s+1) + K_P + K_D s} = \frac{K_P + K_D s}{s^2 + (K_D + 1)s + K_P}$$

So, $\omega_n = \sqrt{K_P}$

$$2\xi\omega_n = K_D + 1$$

$\Rightarrow$ $$\xi = \frac{K_D + 1}{2\sqrt{K_P}}$$

$$\text{Time constant} = \frac{1}{\xi\omega_n} = \frac{2}{K_D + 1}$$

Now, $\text{Time constant} = 1$

$\Rightarrow$ $$\frac{2}{K_D + 1} = 1$$

$\Rightarrow$ $K_D = 1$

Now, $\xi = 0.9$

$\Rightarrow$ $$\frac{K_D + 1}{2\sqrt{K_P}} = 0.9$$

$\Rightarrow$ $$\sqrt{K_P} = \frac{10}{9}$$

$\Rightarrow$ $$K_P = \frac{100}{81}$$

$\Rightarrow$ $K_P = 1.23$

Solution: 2

(i) $$\frac{G}{1+GH} = \frac{\dfrac{K}{s(s+2)}}{1 + \dfrac{Ks}{4(s+2)s}} = \frac{4K}{s(4s+8+K)}$$

$$\frac{G}{1+G} = \frac{4K}{4K + s(4s+8+K)} = \frac{K}{K + s\left(s + 2 + \dfrac{K}{4}\right)}$$

$\therefore$ $$\text{Characteristic equation} = 1 + G(s)$$

$$= s^2 + \left(2 + \frac{K}{4}\right)s + K$$

(ii) For $K = 10$,

$$\text{Characteristic equation} = s^2 + 4.5s + 10$$

Comparing with: $s^2 + 2\xi\omega_n s + \omega_n^2$

We get,

$$\omega_n = \sqrt{10} \text{ rad/sec} = 3.16 \text{ rad/sec}$$
$$2\xi\omega_n = 4.5$$
$$\xi = 0.71$$

(iii) For critical damping,

$$\xi = 1$$

as,

$$\omega_n = \sqrt{K}$$
$$2\xi\omega_n = 2 + \frac{K}{4}$$

$\therefore$

$$2 \times 1 \times \sqrt{K} = 2 + \frac{K}{4}$$
$$K^2 - 48K + 64 = 0$$

$\therefore$

$$K = 1.37 \text{ or } K = 46.62$$

$\Rightarrow$ for critical damping

(iv) For $K = 10$,

$$\omega_n = 3.16 \text{ rad/sec}$$
$$\xi = 0.71$$

For second order system, with unit step input,

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi)$$
$$\omega_d = \sqrt{\omega_n^2 - \xi^2\omega_n^2} = 2.22$$
$$c(t) = 1 - \frac{e^{-2.24t}}{0.7} \sin[2.22t + 45°]$$
$$\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

First peak overshoot time,

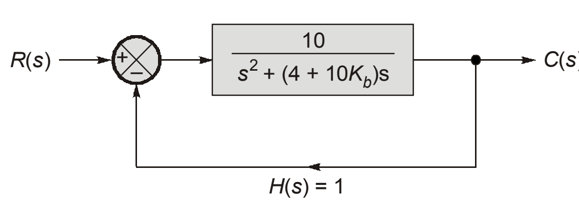
$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{2.22} = 1.415 \text{ sec}$$

Magnitude of first peak overshoot,

$$M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} = 4.21\%$$

Solution: 3

The next redrawn block diagram of the given tachometer feedback control system is,



Now, overall transfer function = $T(s) = \dfrac{C(s)}{R(s)}$

$$\Rightarrow \quad T(s) = \frac{10}{s^2 + (4 + 10K_b)s + 10} \quad \text{...(i)}$$

Given that, ξ = damping ratio = 0.8

From equation (i) characteristics equation is,

$$s^2 + (4 + 10K_b)s + 10 = 0$$

After equating it with the standard 2nd order characteristics equation we get,

$$\omega_n = \sqrt{10} = 3.16 \text{ rad/sec}$$

and $\quad 2\xi\omega_n = 4 + 10K_b$

$\Rightarrow \quad 10K_b = 1.0596$

$$\Rightarrow \quad \boxed{K_b = 0.105}$$

Now, $\quad$ Peak time $(t_p) = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}} = 1.66$ sec.

Peak overshoot $(M_p) = e^{-\pi\varepsilon/\sqrt{1-\varepsilon^2}} = e^{-4.188} = 0.0151$

$\therefore \quad$ %Peak overshoot = 1.51%

Also, damped frequency $(\omega_d) = \omega_n\sqrt{1-\xi^2} = 1.897$ rad/sec

Settling time for ±2% tolerance band is,

$$t_s = \frac{4}{\xi\omega_n} = 1.58 \text{ sec.}$$

Solution : 4

Transfer function , $\quad \dfrac{C(s)}{R(s)} = \dfrac{a}{s^2 + Ks + a} = CLTF$

If $G(s)$ is the forward function for unity feedback system, we have

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = CLTF$$

$$OLTF = \frac{CLTF}{1-CLTF} = \frac{\dfrac{a}{s^2+Ks+a}}{1-\dfrac{a}{s^2+Ks+a}}$$

$$G(s) = \frac{a}{s(s+K)}$$

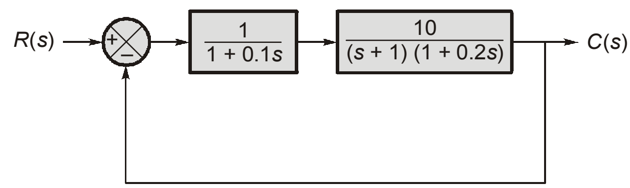
Positional error coefficient $K_p = \underset{s\to 0}{Lt}\, G(s) = \underset{s\to 0}{Lt}\, \dfrac{a}{s(s+K)} = \infty$

Velocity error coefficient $K_v = \underset{s\to 0}{Lt}\, sG(s) = \underset{s\to 0}{Lt}\, s \cdot \dfrac{a}{s(s+K)} = \dfrac{a}{K}$

Acceleration error coefficient $K_a = \underset{s\to 0}{Lt}\, s^2G(s) = \underset{s\to 0}{Lt}\, s^2 \cdot \dfrac{Q}{s(s+K)} = 0$

Solution : 5

We will find out the steady state error due to each input, assuming other input to be zero, and add both to find out total steady state error



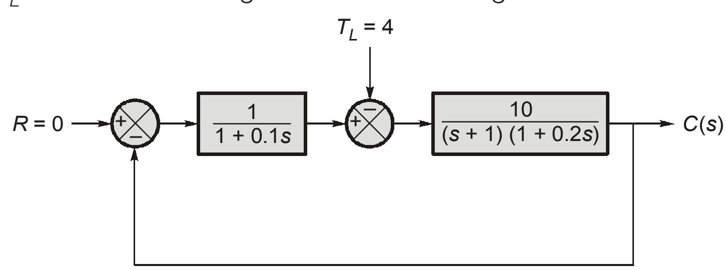
(i) Let $R = 10$, $T_L = 0$. The block diagram will look as follows

$$G(s) = \frac{10}{(1+0.1s)(s+1)(1+0.2s)}$$

$$K_p = \underset{s\to 0}{Lt}\, G(s) = \underset{s\to 0}{Lt} \frac{10}{(1+0.1s)(s+1)(1+0.2s)} = 10$$

Steady state error $$e_{ss} = \frac{A}{1+K_p} = \frac{10}{1+10} = \frac{10}{11} \quad \text{...(i)}$$

(ii) Let $R = 0$ and $T_L = 4$. The block diagram will become as given below.



By applying the Mason's gain formula between T_L and $E(s)$.

$$\frac{E(s)}{T_L(s)} = \frac{(-1)\left[\frac{10}{(s+1)(1+0.2s)}\right](-1)}{1+\left[\frac{10}{(s+1)(1+0.2s)}\right]\left[\frac{1}{1+0.1s}\right]}$$

$$\frac{E(s)}{T_L(s)} = \frac{10(1+0.1s)}{(s+1)(1+0.2s)(1+0.1s)+10}$$

$$e_{ss} = \underset{s\to 0}{Lt}\, sE(s)$$

$$= \underset{s\to 0}{Lt}\, s\left[\frac{10(1+0.1s)}{(s+1)(1+0.2s)(1+0.1s)+10}\right]\times\frac{4}{s} = \frac{40}{11} \quad \text{...(2)}$$

$$\text{Total error} = (1) + (2) = \frac{10}{11}+\frac{40}{11} = \frac{50}{11} = 4.54$$

Solution : 6

$$\frac{C(s)}{R(s)} = \frac{\frac{15}{(s+1)(s+3)}}{1+\frac{15}{(s+1)(s+3)}} = \frac{15}{s^2+4s+18}$$

Hence, characteristic equation is,

$$s^2 + 4s + 18 = 0$$

$$\therefore \quad \omega_n = \sqrt{18} = 4.2426 \text{ rad/sec}$$

$$2\xi\omega_n = 4$$

$$\xi = 0.4714$$

Now, time at which $c(t)$ will experience maxima is given by $t_p = \dfrac{n\pi}{\omega_d}$.

where $n = 1$ ...1st overshoot

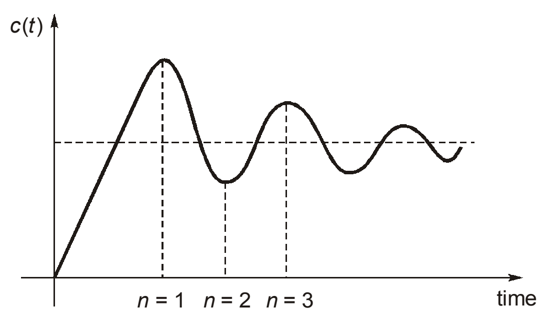
$n = 2$ ...1st undershoot

$n = 3$ ...2nd overshoot

t for 1st undershoot $= \dfrac{2\pi}{\omega_d}$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 3.74166 \text{ rad/sec}$$

This can be shown as follows.



$\therefore$ t for 1st undershoot $= \dfrac{2\pi}{3.74166} = 1.6792$ sec

now time period of oscillations is related to ω_d by relation, $\omega_d = \dfrac{2\pi}{T}$.

Where T = Time period,

$$T = \frac{2\pi}{\omega_d} = \frac{2\pi}{3.74166} = 1.6792 \text{ sec/cycle}$$

For 1st cycle, output will take 1.6792 sec.

Now the time for 1 cycle is known and if it is known to us that what is the time required by the system to achieve steady state, we can find how many cycles output will perform before reaching steady state.

$$\therefore \quad t_r = \frac{4}{\xi\omega_n} = 2 \text{ sec}$$

So, 1.6792 sec for one cycle, how many cycles output will perform in 2 sec.

$$\therefore \quad \text{Total number of cycles} = \frac{2}{1.6792} = 1.191$$

Output will perform 1.191 cycles before reaching the steady state.

■■■■

4 Stability

LEVEL 1 Objective Solutions

1.	(b)
2.	(2)
3.	(b)
4.	(b)
5.	(c)
6.	(d)
7.	(c)
8.	(b)
9.	(d)
10.	(b)
11.	(c)
12.	(d)
13.	(b)
14.	(b)
15.	(c)

LEVEL 2 Objective Solutions

16.	(1.41)
17.	(1)
18.	(2.1)
19.	(d)
20.	(c)
21.	(a)
22.	(d)
23.	(d)
24.	(b)
25.	(a)
26.	(b)
27.	(d)
28.	(25)
29.	(2)

LEVEL 3 Conventional Solutions

Solution: 1

Characteristic equation:

$\Rightarrow$ $s(s + 1)(s + 2)(s + 3) + K(s + a) = 0$

$\Rightarrow$ $s(s + 1)(s^2 + 5s + 6) + K(s + a) = 0$

$\Rightarrow$ $s[s^3 + 5s^2 + 6s + s^2 + 5s + 6] + K(s + a) = 0$

$\Rightarrow$ $s[s^3 + 6s^2 + 11s + 6] + K(s + a) = 0$

$\Rightarrow$ $s^4 + 6s^3 + 11s^2 + (6 + K)s + Ka = 0$

$$\begin{array}{c|ccc} s^4 & 1 & 11 & aK \\ s^3 & 6 & K+6 & \\ s^2 & \dfrac{66-(K+6)}{6} & aK & \\ s^1 & (K+6) & \dfrac{6aK}{\left(\dfrac{66-(K+6)}{6}\right)} & \\ s^0 & aK & & \end{array}$$

From s^2 now

$$\frac{66-(K+6)}{6} > 0$$

$\Rightarrow$ $K < 60$

From s^0 now

$$aK > 0$$

$\Rightarrow$ $K > 0$ [As given $a > 0$] so, range of K ; $\boxed{0 < K < 60}$

From s^1 now

$$(K+6) - \frac{36aK}{60-K} > 0$$

$\Rightarrow$ $K + 6 + \dfrac{36aK}{K-60} > 0$

$\Rightarrow$ $K + 6 > \dfrac{36aK}{60-K}$

$\Rightarrow$ $\dfrac{(K+6)(60-K)}{36K} > a$

$\Rightarrow$ $a < \dfrac{(K+6)(60-K)}{36K}$

$\Rightarrow$ $a < \dfrac{(60K - K^2 + 360 - 6K)}{36K}$

$\Rightarrow$ $a < \dfrac{(54K - K^2 + 360)}{36K}$

$\Rightarrow$ $a < \dfrac{1}{36}\left(54 - K + \dfrac{360}{K}\right)$

Solution: 2

Given that, $$G(s) = \frac{K(s+2)}{s^3 + Ps^2 + 3s + 2} \quad \text{...(i)}$$

Since the system is critically stable so,

$$\xi = 0$$

From equation (i), the characteristics equation is,

$$s^3 + Ps^2 + 3s + 2 + Ks + 2K = 0$$

$$\Rightarrow \quad s^3 + Ps^2 + (3 + k)s + (2 + 2k) = 0 \quad \text{...(ii)}$$

So, Routhian-array constructed as,

$$\begin{array}{c|cc} s^3 & 1 & (3+K) \\ s^2 & P & (2+2K) \\ s^1 & \left(\dfrac{3P+KP-2-2K}{P}\right) & \\ s^0 & 2+2K & \end{array}$$

The system have a sustained oscillations with a frequency of 2.5 rad/sec if s^1-row is zero i.e.

$$(3 + K)P = 2 + 2K$$

$$\Rightarrow \quad 3P - 2 = 2K - KP \quad \text{...(iii)}$$

So, Auxiliary equation along s^2-row is,

$$Ps^2 + 2 + 2K = 0$$

$$\Rightarrow \quad P(j\omega)^2 = -(2 + 2K)$$

$$\Rightarrow \quad 6.25P = 2 + 2K$$

$$\therefore \quad P = \frac{2+2K}{6.25} \quad \text{...(iv)}$$

Now from equation (iii),

$$\Rightarrow \quad 3 \times \left(\frac{2+2K}{6.25}\right) - 2 = 2K - \frac{K(2+2K)}{6.25}$$

$$\Rightarrow \quad -6.5 + 6K - 12.5K + 2K + 2K^2 = 0$$

$$\Rightarrow \quad 2K^2 - 4.5\,K - 6.5 = 0$$

$$\therefore \quad K = 3.25 \text{ and } -1$$

$$\text{Gain } (K) = +\text{ve so, } \boxed{K = 3.25}$$

Now from equation (iv),

$$\boxed{P = 1.36}$$

Solution : 3

$$G(s) = K\frac{(s+2)(s+1)}{(s-1)(s+0.1)}$$

$$H(s) = 1 \quad \text{(unity feedback)}$$

$$1 + G(s)\,H(s) = 1 + \frac{K(s+2)(s+1)}{(s-1)(s+0.1)}$$

$$\text{Characteristic equation} = (s-1)(s+0.1) + K(s+2)(s+1) = 0$$

$$s^2 - 0.9s - 0.1 + K(s^2 + 3s + 2) = 0$$

$$(K + 1)s^2 + s\,(3K - 0.9) + (2K - 0.1) = 0$$

Using Routh's tabular form:

$$\begin{array}{c|cc} s^2 & (K+1) & (2K-0.1) \\ s & 3K-0.9 & \\ 1 & 2K-0.1 & \end{array}$$

For 0 poles in RHS of s-plane

All the coefficients in 1st column must be of same sign

$$\therefore \quad K+1>0 \Rightarrow K>-1$$

$$3K-0.9>0 \Rightarrow K>0.3$$

$$2K-0.1>0 \Rightarrow K>0.05$$

$$\therefore \quad \boxed{K>0.3}$$

For 1 pole in RHS of s-plane there should be 1 sign change in first column.

$$\therefore \quad K>-1$$

$$K<+0.3$$

$$K<+0.05$$

$$\therefore \quad 0.05>K>-1$$

For 2 pole there should be 2 sign change

$$K<-1$$

$$K<0.3$$

$$K>0.05$$

$$0.3>K>0.05$$

Solution : 4

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 11 \\ s^4 & 1 & 2 & 10 \\ s^3 & 0=\dfrac{1\times 2-1\times 2}{1} & 1=\dfrac{1\times 11-1\times 10}{1} & 0 \\ s^2 & & & \end{array}$$

While forming the Routh array as above the third element in the first column is zero and thus the Routh criterion fails at this stage. The difficulty is solved if zero in third row of the first column is replaced by a symbol ε and Routh array is formed as follows:

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 11 \\ s^4 & 1 & 2 & 10 \\ s^3 & \varepsilon & 1 & 0 \\ s^2 & \underset{\varepsilon\to 0}{Lt}\dfrac{\varepsilon\times 2-1\times 1}{\varepsilon}=-\infty & 10 & 0 \\ s^1 & \underset{\varepsilon\to 0}{Lt}\left(1-\dfrac{10\varepsilon^2}{2\varepsilon-1}\right)=1 & 0 & 0 \\ s^0 & & & \end{array}$$

The limits of fourth and fifth element in the first column as $\varepsilon \to 0$ from positive side are $-\infty$ and 1 respectively, indicating two sign changes, therefore, the system is unstable and the number of roots with positive real part of the characteristic equation is 2.

Solution : 5

$$\begin{array}{c|ccc} s^4 & 1 & 15 & K \\ s^3 & 20 & 2 & \\ s^2 & 14.9 & K & \\ s^1 & \dfrac{29.8-20K}{149} & & \\ s^0 & K & & \end{array}$$

(a) For stability

$$K > 0$$
$$29.8 - 20\,K > 0$$
$$K < 1.49$$

(b) For marginally stable $K = 1.49$

Auxiliary equation $A(s) = 14.9s^2 + 1.49$

$$14.9s^2 = -1.49$$
$$\Rightarrow \quad s^2 = 0.1$$
$$s = \pm j0.316$$
$$\omega = 0.316 \text{ rad/sec}$$

∴ Frequency of sustained oscillation = 0.316 rad/sec

Solution : 6

The characteristic equation of the system is

$$1 + G(s)\,H(s) = 0$$

$$1+\frac{K(s+5)}{s(1+Ts)(1+2s)} = 0$$

$$\therefore \quad s(1 + Ts)\,(1 + 2s) + K(s + 5) = 0$$
$$2Ts^3 + (T + 2)s^2 + s(k + 1) + 5k = 0$$

∴ Routh's array is

$$\begin{array}{c|cc} s^3 & 2T & K+1 \\ s^2 & T+2 & 5K \\ s^1 & \dfrac{(T+2)(K+1)-2T\times 5K}{(T+2)} & 0 \\ s^0 & 5K & \end{array}$$

From last row $5K > 0$

K must be positive

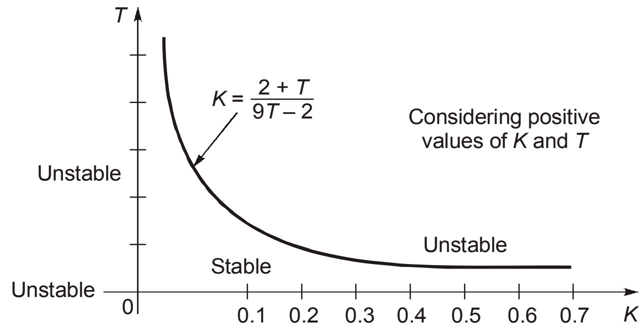
From row of s'

$$(T + 2)\,(K + 1) - 10KT > 0$$
$$\therefore \quad 2K + T + 2 - 9KT > 0$$

Limiting value of $\quad 2K + T + 2 - 9\,KT > 0$

i.e., $\quad K < \dfrac{2+T}{9T-2}$

∴ Region in which a closed loop system is stable is



Solution : 7

$$\begin{array}{c|cccc} s^6 & 1 & 4 & 5 & 2 \\ s^5 & 3 & 6 & 3 & 0 \\ s^4 & 2 & 4 & 2 & 0 \\ s^3 & 0 & 0 & 0 & 0 \end{array}$$

Row of zeros

$$A(s) = 2s^4 + 4s^2 + 2 = 0$$

i.e.,
$$s^4 + 2s^2 + 1 = 0$$

$$\frac{dA(s)}{ds} = 4s^3 + 4s$$

$$\begin{array}{c|cccc} s^6 & 1 & 4 & 5 & 2 \\ s^5 & 3 & 6 & 3 & 0 \\ s^4 & 2 & 4 & 2 & 0 \\ s^3 & 4 & 4 & 0 & 0 \\ s^2 & 2 & 2 & 0 & 0 \\ s^1 & 0 & 0 & 0 & 0 \end{array}$$

Row of zeros again,

$$A'(s) = 2s^2 + 2 = 0$$

$$\frac{dA'(s)}{ds} = 4s = 0$$

$$\begin{array}{c|cccc} s^6 & 1 & 4 & 5 & 2 \\ s^5 & 3 & 6 & 3 & 0 \\ s^4 & 2 & 4 & 2 & 0 \\ s^3 & 4 & 4 & 0 & 0 \\ s^2 & 2 & 2 & 0 & 0 \\ s^1 & 4 & 0 & 0 & 0 \\ s^0 & 2 & 0 & 0 & 0 \end{array}$$

No sign change in the 1st column.

∴ Number of RHP = 0

Row of zeros occur twice implies the auxiliary equation roots are multiple or repeated.
Number of $j\omega P = 4$ (repeated)
Number of LHP = 2
As there are repeated roots on imaginary axis, system is unstable.

■■■■

5 Root Locus Technique

LEVEL 1 Objective Solutions

1.	(c)
2.	(c)
3.	(b)
4.	(b)
5.	(d)
6.	(b)
7.	(c)
8.	(d)
9.	(a)
10.	(c)
11.	(c)
12.	(c)
13.	(b)
14.	(b)
15.	(d)
16.	(a)
17.	(c)

LEVEL 2 Objective Solutions

18.	(c)
19.	(a)
20.	(d)
21.	(d)
22.	(c)
23.	(c)
24.	(a)
25.	(d)
26.	(c)
27.	(d)
28.	(b)
29.	(b)
30.	(c)
31.	(b)

LEVEL 3 Conventional Solutions

Solution: 1

- Root-locus of any control system is fully depends upon Open Loop Transfer Function (OLTF). Given characteristic equation is,

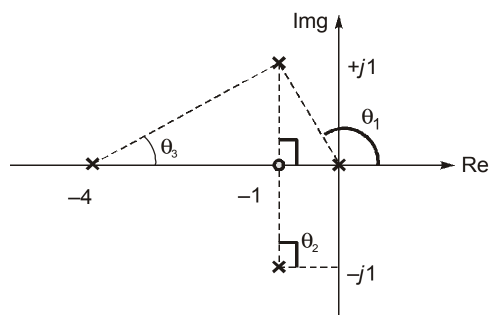
$$\Rightarrow \quad s(s+4)(s^2+2s+2)+K(s+1)=0$$

$$\Rightarrow \quad 1+\frac{K(s+1)}{s(s+4)(s^2+2s+2)}=0$$

Comparing with $1 + G(s)\,H(s) = 0$, the open-loop transfer function,

$$G(s)\,H(s)=\frac{K(s+1)}{s(s+4)(s^2+2s+2)}$$

The open–loop poles are located at $s = 0, -4, (-1 + j1)$ and $(-1 - j1)$. So, $\boldsymbol{n = 4}$.
The open-loop zero is located at $\boldsymbol{s = -1}$. So, $\boldsymbol{m = 1}$. The pole-zero configuration is shown below.



Angle subtend by zeros = $\phi_Z = 90°$
Angle subtend by poles = $\phi_P = \theta_1 + \theta_2 + \theta_3$
$\therefore$ Angle of departure = $\phi_D = 180° - \{\phi_P - \phi_Z\}$ ...(i)
Root locus on real axis is in the region $\boldsymbol{-1 < s < 0}$ and $\boldsymbol{s < -4}$.
Number of asymptotes = $n - m = 4 - 1 = 3$

$$\text{Angles of asymptotes, } \phi_A = \frac{(2q+1)180°}{n-m};\ q = 0, 1, 2$$

Putting values

$$\phi_A = \frac{(2q+1)180°}{3};\ q = 0, 1, 2$$

$$\phi_A = 60°, 180°, 300°$$

$$\text{Centroid} = \frac{(\text{sum of real parts of poles} - \text{sum of real parts of zeros})}{n-m}$$

$$\text{Centroid} = \frac{\{0-4-1-1-(-1)\}}{3}$$

$$\text{Centroid} = \frac{-5}{3} = -1.67$$

Angle of departure at $s = -1 + j$:

$$\phi_D = 180° - (\phi_P - \phi_Z) \approx 180° + (\phi_Z - \phi_P)$$

$$GH'(s=-1+j) = \frac{K(-1+j+1)}{(-1+j)(-1+j+4)(-1+j+1+j)} = \frac{K}{2(-1+j)(3+j)}$$

$$\angle GH'(s=-1+j) = -135° - \tan^{-1}\frac{1}{3} = -135° - 18.435° = -153.435°$$

$$\phi_D = 180° - 153.435° = 26.565°$$

Since the root locus is symmetrical about the real axis, the angle of departure at $s = -1 - j$ is $-26.565°$.

⇒ The intercepts of the asymptotes is nothing but intersection of Root Locus on the $j\omega$ axis and it can be determined by the use of the Routh criterion.

Here characteristic equation is,

$$s(s+4)(s^2+2s+2) + K(s+1) = 0$$

or

$$s^4 + 6s^3 + 10s^2 + (K+8)s + K = 0$$

s^4	1	10	K
s^3	6	$K+8$	
s^2	$\frac{60-K-8}{6}$	K	
s^1	$\dfrac{\frac{(52-K)(K+8)}{6} - 6K}{(52-K)/6}$	0	
s^0	K		

Critical value of K is given by

$$\frac{(52-K)(K+8)}{6} - 6K = 0$$

$$(52-K)(K+8) - 36K = 0$$

$$416 + 44K - K^2 - 36K = 0$$

$$K^2 - 8K - 416 = 0$$

$$K = \frac{8 \pm \sqrt{64 + 4 \times 416}}{2} = 8 + \frac{41.57}{2} = 24.785$$

[Negative value of K has been discarded as $K > 0$ according to Routh array.]

Putting array along s^2 row,

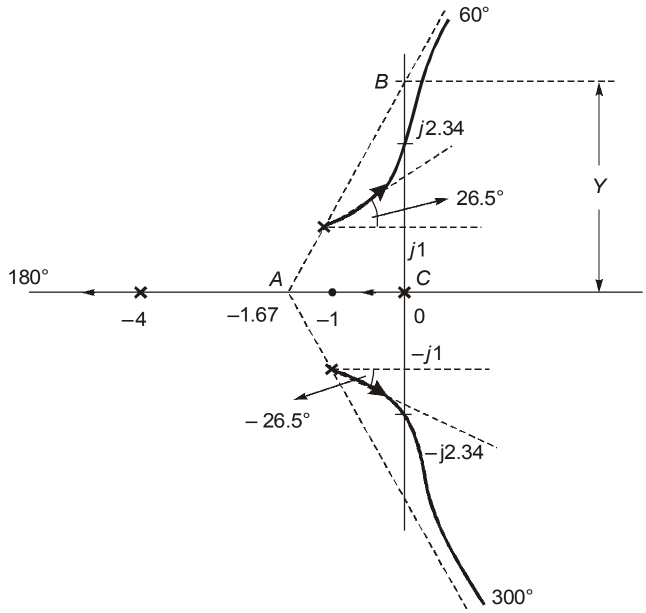
$$\left(\frac{52-24.785}{6}\right)s^2 + 24.785 = 0$$

$$4.536\, s^2 + 24.785 = 0$$

$$s^2 = \frac{-24.785}{4.536} = -5.464$$

$$s = \pm j2.34$$

The complete root locii is shown below:



From triangle *ABC*, we have

$$\tan 60^\circ = \frac{Y}{1.67}$$

$$\Rightarrow \qquad \boxed{Y = 2.89}$$

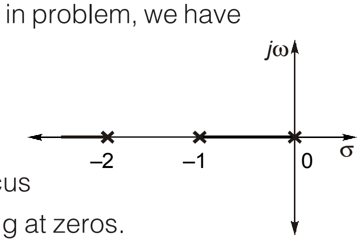
Solution: 2

Characteristic equation is given as

$$1 + G(s)\,H(s) = 0$$

On comparing this characteristic equation with the equation given in problem, we have

$$G(s)\,H(s) = \frac{K}{s(s+1)(s+2)}$$



P = Number of open loop poles

= 3 = number of branches on root locus

Z = 0 = Number of branches terminating at zeros.

Angle of Asymptotes: The $P - Z$ branches terminating at infinity will go along certain straight lines.

Number of asymptotes = $P - Z$

= $3 - 0 = 3$

$$\theta = \frac{180^\circ(2q+1)}{P-Z} \qquad q = 0.\ 1,\ 2\ \ldots$$

$$\theta_1 = \frac{180 \times (2 \times 0 + 1)}{3} = 60^\circ$$

$$\theta_2 = \frac{180^\circ(2 \times 1 + 1)}{3} = 180^\circ$$

$$\theta_3 = \frac{180^\circ(2 \times 2 + 1)}{3} = 300^\circ$$

Centroid: It is the intersection point of the asymptotes on the real axis. It may or may not be a part of root locus.

$$\text{Centroid} = \frac{\Sigma \text{ Real part of open loop poles } - \Sigma \text{ Real part of open loop zeros}}{P - Z} = \frac{0-1-2}{3} = -1$$

Centroid → (–1, 0)

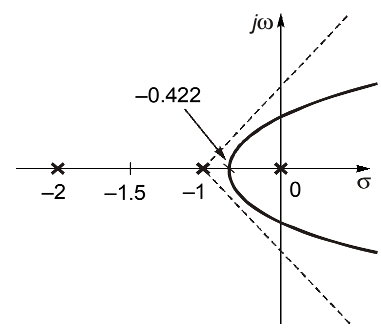
Break-away or break-in points: These are those points whose multiple roots of the characteristic equation occur.

$$s(s^2 + 3s + 2) + K = 0$$

$$K = -(s^3 + 3s^2 + 2s)$$

$$\frac{dK}{ds} = -(3s^2 + 6s + 2) = 0$$

$$s = -0.422, -1.577$$



Now verify the valid break-away point

$$K = 0.234 \text{ (valid) at } s = -0.422$$

$$K = \text{negative (not valid) at } s = -1.577$$

Solution : 3

Characteristic equation : $1 + G(s)H(s) = 0$

$$1 + \frac{K}{s(s+1)(s+4)} = 0$$

i.e. $s^3 + 5s^2 + 4s + K = 0$

$$K = -s^3 + 5s^2 - 4s$$

$$\frac{dk}{ds} = -3s^3 + 10s + 4 = 0$$

∴ $$\text{Breakaway points} = \frac{-10 \pm \sqrt{100 - 4 \times 4 \times 3}}{2 \times 3}$$

$$= -0.46, -2.86$$

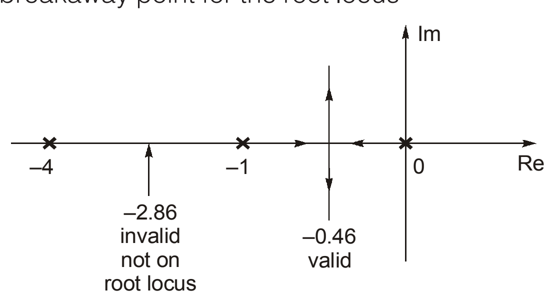
Substituting in expression of K

For $s = -0.46$, $K = +0.8793$

For $s = -2.86$, $K = -6.064$

∴ For $s = -0.46$, K is positive

Hence, $s = -0.46$ is valid breakaway point for the root locus



Solution : 4

Poles are 0, 0, –9 and zero is –1

Therefore, $P = 3$, $Z = 1$

$$\text{Number of asymptotes} = |P - Z| = 2$$

$$\text{Number of RLD branches} = 3$$

$$\text{Centroid, } \sigma = \frac{-9-(-1)}{3-1} = -4$$

$$\text{Angle of asymptotes, } \theta_i = \frac{(2l+1)180°}{P-Z};\ l = 0 \text{ and } 1$$

$$\theta_0 = 90°,\ \theta_1 = -90° \text{ or } 270°$$

Break points:

$$1 + G(s)\,H(s) = 0$$

$$1 + \frac{K(s+1)}{s^2(s+9)} = 0 \Rightarrow K = \frac{-s^2(s+9)}{(s+1)}$$

$$\frac{dK}{ds} = 0$$

$$\frac{d}{ds}\left(\frac{-s^2(s+9)}{s+1}\right) = 0$$

$$-[(s^3 + 9s^2)(1) - (s+1)(3s^2 + 18s)] = 0$$

$$s^3 + 9s^2 - 3s^3 - 3s^2 - 18s^2 - 18s = 0$$

$$-2s^3 - 12s^2 - 18s = 0$$

$$\Rightarrow \quad s^3 + 6s^2 + 9s = 0$$

$$s(s+3)(s+3) = 0$$

$s = 0$ and $s = -3$ are the break points

$$\text{CE :} \quad 1 + \frac{K(s+1)}{s^3 + 9s^2} = 0$$

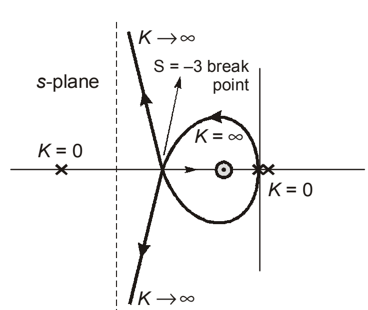
$$s^3 + 9s^2 + K(s+1) = 0$$

At $s = 0$;
$$0 + 0 + K = 0$$
$$K = 0 \text{ i.e. +ve}$$

At $s = -3$;
$$(-3)^3 + 9(-3)^2 + K(-3+1) = 0$$
$$-2K + 81 - 27 = 0$$

$K = 27$ i.e. positive therefore at $s = 0$ and $s = -3$, the value of K is positive, therefore the break points are valid.

The RLD is shown below,



Solution : 5

Open loop poles are located at $s = 1$, $s = -2 + j\sqrt{3}$ and $-2 - j\sqrt{3}$. A root locus exists on the real axis between point $s = 1$ and $s = -\infty$. The asymptotes of the roof-locus branches are found as follows:

Angle of asymptotes $= \dfrac{\pm 180°(2l+1)}{3} = 60°, -60°, 180°$

The intersection of the asymptotes on the real axis is obtained as

$$\sigma = \frac{1-2-2}{3} = -1$$

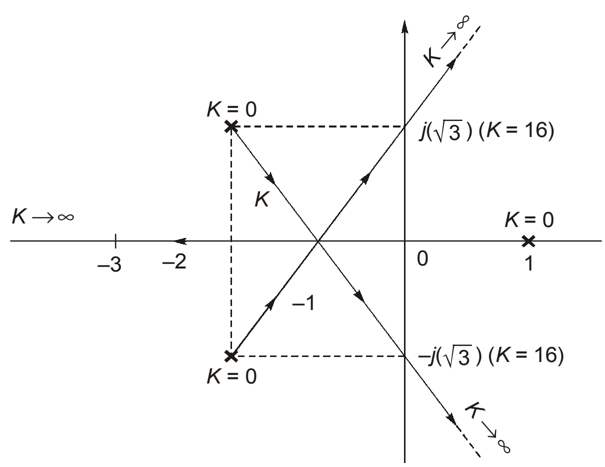
The breakaway and break-in points can be located from $\dfrac{dk}{ds} = 0$

Since,
$$K = -(s-1)\left(s^2+4s+7\right)$$
$$= -\left(s^3+3s^2+3s-7\right)$$

We have
$$\frac{dk}{ds} = -(3s^2+6s+3) = 0$$

Which yields $(s+1)^2 = 0$

$\Rightarrow$ $s = -1$ is a break point



Point of intersection of the RLD with respect to imaginary axis.

CE : $(s-1)(s^2+4s+7) + K = 0$

$$s^3 + 3s^2 + 3s - 7 + K = 0$$

The Routh's tabulation for above equation is

s^3	1	3
s^2	3	$K-7$
s^1	$\dfrac{9-K+7}{3}$	
s^0	$K-7$	

$$9 + 7 - K = 0$$
$$K = 16$$

Auxiliary equation: $\quad AE = 3s^2 + K - 7 = 0$

$$3s^2 + 16 - 7 = 0$$
$$s = \pm j\sqrt{3}$$

■■■■

6 Frequency Response Analysis

LEVEL 1 Objective Solutions

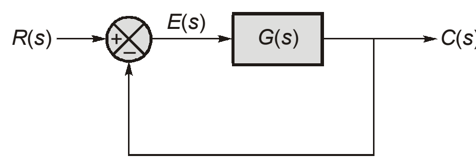
1.	(a)
2.	(10)
3.	(2.50)
4.	(d)
5.	(a)
6.	(d)
7.	(b)
8.	(d)
9.	(c)
10.	(a)
11.	(c)
12.	(a)
13.	(a)
14.	(a)
15.	(b)
16.	(c)
17.	(b)
18.	(c)
19.	(b)
20.	(c)
21.	(c)
22.	(a)

LEVEL 2 Objective Solutions

23.	(c)
24.	(b)
25.	(b)
26.	(0.5)
27.	(d)
28.	(d)
29.	(a)
30.	(c)
31.	(a)
32.	(c)
33.	(c)
34.	(c)
35.	(a)
36.	(c)
37.	(d)
38.	(b)
39.	(b)
40.	(c)
41.	(a)
42.	(b)
43.	(b)

LEVEL 3 Conventional Solutions

Solution: 1



$$G(s) = \frac{K}{s^2(1+sT)}$$

$$G(j\omega) = \frac{K}{-\omega^2(1+j\omega T)} = \frac{-K}{\omega^2(1+\omega^2T^2)} + \frac{jKT}{\omega(1+\omega^2T^2)}$$

$$|G(j\omega)| = \frac{K}{\omega^2\sqrt{1+\omega^2T^2}}$$

$$\angle G(j\omega) = \pi - \tan^{-1}\omega T$$

$$\lim_{\in\to 0} G(\in e^{j\theta}) = \frac{K}{\in^2 e^{j2\theta}} = \frac{K}{\in^2}e^{-j2\theta}$$

$$\theta : -\frac{\pi}{2} \to 0 \to +\frac{\pi}{2}$$

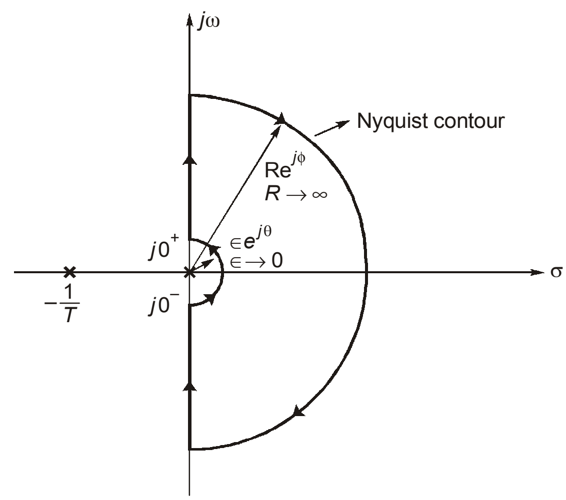
$$\angle G(\in \omega^{j\theta}) : \pi \to 0 \to -\pi$$

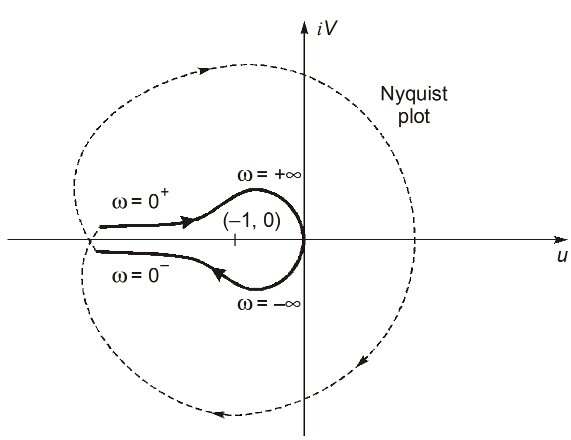
$$\lim_{R\to\infty} G(Re^{j\phi}) = \lim_{R\to\infty}\frac{K}{R^2e^{j2\phi}(1+Re^{j\phi}T)}$$

$$= \lim_{R\to\infty}\frac{K}{R^3e^{j3\phi}T} = 0$$

$$\phi : \frac{\pi}{2} \to 0 \to -\frac{\pi}{2}$$

$$\angle G(Re^{j\phi}) : -\frac{3\pi}{2} \to 0 \to \frac{3\pi}{2}$$

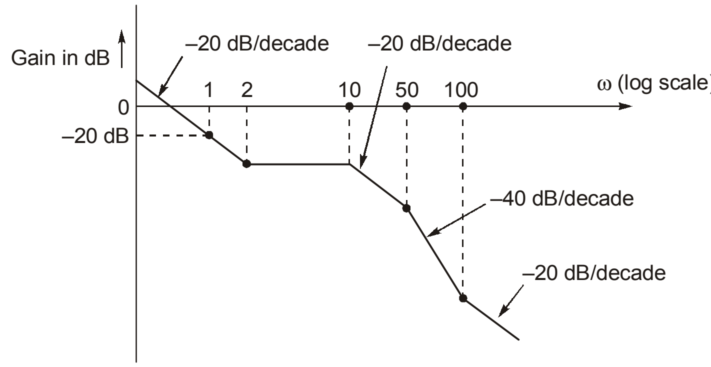




This system is unstable as the Nyquist plot encircles (–1, 0) twice in clockwise direction.

Solution: 2

Initial slope –6 dB/octave or –20 dB/decade represents pole at origin



Corner frequencies ω = 2, 10, 50, 100. Where ω = 2, 100 → zeros, ω = 10, 50 → Poles

$$G(s) = \frac{K\left(\frac{s}{2}+1\right)\left(\frac{s}{100}+1\right)}{s\left(\frac{s}{10}+1\right)\left(\frac{s}{50}+1\right)}$$

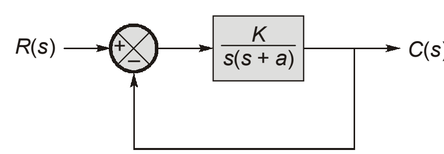
At $\omega = 1$ $|G(j\omega)| = -20\text{ dB}$, so $-20 = 20\log_{10} K - 20\log_{10}\omega$ at $\omega = 1$

$$K = \frac{1}{10}$$

$\Rightarrow$
$$G(s) = \frac{\left(1+\frac{s}{2}\right)\left(1+\frac{s}{100}\right)}{(10s)\left(1+\frac{s}{10}\right)\left(1+\frac{s}{50}\right)}$$

Solution: 3

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+a)}}{1+\frac{K}{s(s+a)}} = \frac{K}{s^2+as+K}$$



Comparing the above equation with the standard second order equation as = $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

On comparing $\omega_n^2 = K$

$$2\xi\omega_n = a$$

Given $M_r = 1.04 = \frac{1}{2\xi\sqrt{1-\xi^2}}$

$$\xi^2(1-\xi^2) = 0.231139$$

$$\xi^4 - \xi^2 + 0.231139 = 0$$

$$\xi = 0.622, 0.7983$$

For the real values

$$\xi < \frac{1}{\sqrt{2}}$$

So, the appropriate value of $\xi = 0.622$

Now, $$\omega_r = 11.55 \text{ rad/sec} = \omega_n\sqrt{1-2\xi^2}$$

$$\omega_n = 24.283 \text{ rad/sec}$$

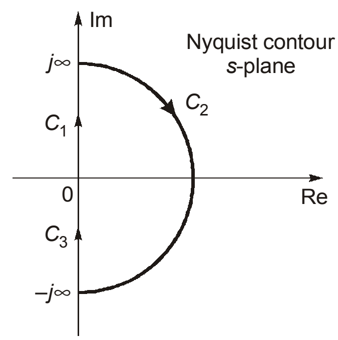
Now $$K = \omega_n^2$$

$$\boxed{K = 589.66}$$

$$2\xi\omega_n = a$$

$$\boxed{a = 30.21}$$

Solution : 4



Mapping sections 'C_1' of the Nyquist contour.
Substituting $s = j\omega$, $0 \le \omega \le \infty$.

$$G(j\omega)\,H(j\omega) = \frac{8j\omega}{(j\omega-1)(j\omega-2)}$$

$$= \frac{8\omega}{\sqrt{(\omega^2+1)\,(\omega^2+4)}} \angle 90° - \left(180° - \tan^{-1}(\omega) + 180° - \tan^{-1}\left(\frac{\omega}{2}\right)\right)$$

$$\omega = 0 \qquad 0\angle 90°$$

$$\omega = \infty \qquad 0\angle -90°$$

Point of intersection of the Nyquist plot will respect to negative real axis.

$$\angle G(j\omega)\,H(j\omega) = -180° \text{ at } \omega = \omega_{pc}$$

$$90° - 180° + \tan^{-1}\omega_{pc} - 180° + \tan^{-1}\frac{\omega_{pc}}{2} = -180°$$

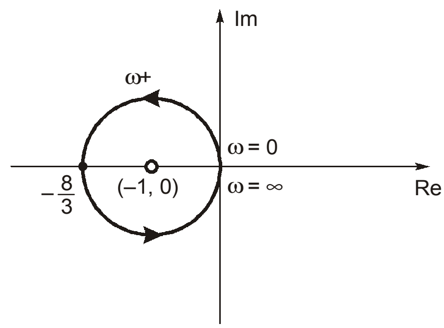
$$\tan^{-1}\omega_{pc} + \tan^{-1}\frac{\omega_{pc}}{2} = 90°$$

$$\frac{\omega_{pc} + \dfrac{\omega_{pc}}{2}}{1 - \dfrac{\omega_{pc}^2}{2}} = \infty \qquad \Rightarrow \omega_{pc}^2 = 2$$

$$\omega_{pc} = \sqrt{2} \text{ rad/sec}$$

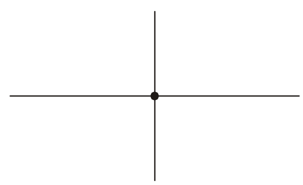
$$\text{POI} = a = |G(j\omega)|_{\omega=pc} = \frac{8\,\omega_{pc}}{\sqrt{\left(\omega_{pc}^2+1\right)\,\left(\omega_{pc}^2+4\right)}}$$

$$a = \frac{8\sqrt{2}}{\sqrt{(2+1)\,(2+4)}} = \frac{8\sqrt{2}}{\sqrt{18}} = \frac{8}{3}$$



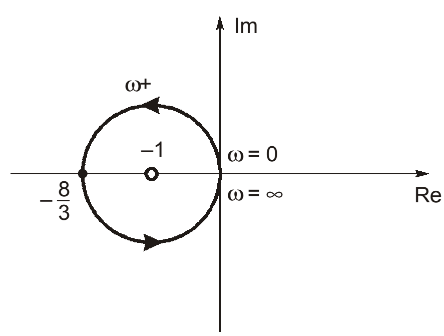
Mapping section C_2 of the Nyquist contour substitute $s = \lim\limits_{R\to\infty} Re^{j\theta}$; $90^\circ \geq \theta \geq -90^\circ$

$$\lim_{R\to\infty} \left| G(Re^{j\theta})\, H(Re^{j\theta}) \right| = \lim_{R\to\infty} \frac{8Re^{j\theta}}{(Re^{j\theta} - 1)\,(Re^{j\theta} - 2)} = 0$$

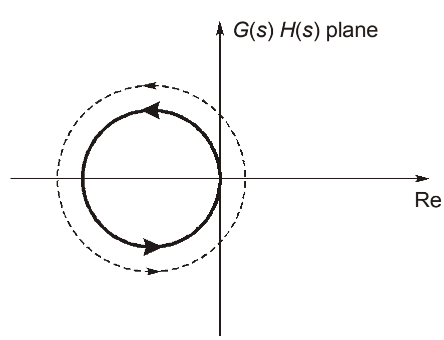


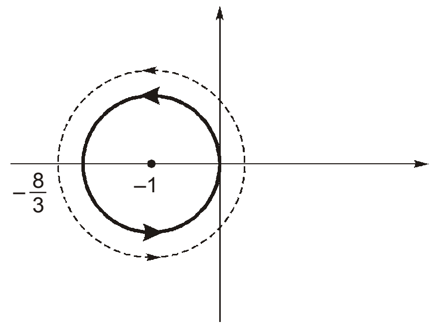
Mapping section C_3 of Nyquist contour substitute $s = j\omega$; $-\infty \leq \omega \leq 0$

It is shown below,



Combining all the above section





$$N = P - Z,\ P = 2,\ N = 2$$
$$Z = P - N = 0\ ;\ \text{stable}$$

Solution : 5

$$G(s) = \frac{e^{-sT}}{s(0.5s+1)(0.2s+1)}$$

Put $s = j\omega$

$$\therefore \quad G(j\omega) = \frac{e^{-j\omega T}}{j\omega(j0.5\omega+1)(j0.2\omega+2)}$$

To determine gain crossover frequency ω_{gc} put $|G(j\omega_{gc})| = 1$

$$\therefore \quad \left|\frac{e^{-j\omega_{gc}t}}{j\omega_{gc}(j0.5\omega_{gc}+1)(j0.2\omega_{gc}+1)}\right| = 1$$

$$\therefore \quad \left|\frac{1}{\omega_{gc}\sqrt{(0.5\omega_{gc})^2+1}\ \sqrt{(0.2\omega_{gc})^2+1}}\right| = 1$$

or
$$\frac{1}{\omega_{gc}^2(0.25\omega_{gc}^2+1)(0.04\omega_{gc}^2+1)} = 1$$

$$\therefore \quad \omega_{gc}^2(0.25\omega_{gc}^2+1)(0.04\omega_{gc}^2+1) - 1 = 0$$

Put $\omega_{gc}^2 = X$

By solving above equation with consideration of X instead of ω_{gc}^2 ,

$$\therefore \quad \omega_{gc} = \sqrt{0.8062} = 0.897 \text{ rad/sec}$$

$$\text{Phase margin} = 180° + \angle G_1(j\omega_1)$$
$$= 180° + (-90° - \tan^{-1}0.5\omega_{gc} - \tan^{-1}0.2\omega_{gc})$$
$$= 90° - \tan^{-1} 0.5 \times 0.897 - \tan^{-1} 0.2 \times 897$$
$$= 90° - 24.156° - 10.17° = 55.6°$$

The closed loop system is stable with a phase margin is 55.6°

(ii) To make the system marginally stable, the phase angle to be provided by the time delay element – 55.6°.

The phase contributed with the insertion of time delay element ($e^{-j\omega T}$) is

$$\frac{-\omega_{gc} T \times 180}{\pi} = -55.6$$

$$\therefore \quad T = \frac{55.6 \times \pi}{\omega_{gc} \times 180}$$

$$\omega_{gc} = 0.897 \text{ rad/sec}$$

$$T = \frac{55.6 \times \pi}{0.897 \times 180} = 1.08 \text{ sec}$$

Solution : 6

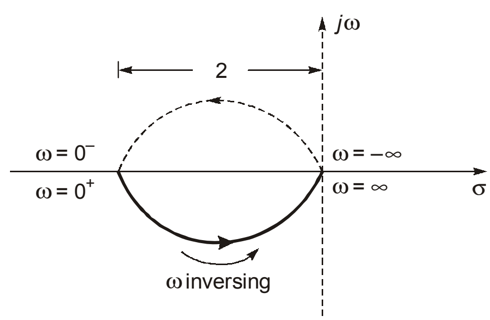
$$G(s)\,H(s) = \frac{s+2}{(s+1)(s-1)} = \frac{j\omega + 2}{(j\omega+1)(j\omega-1)}$$

$$= \sqrt{\frac{\omega^2+4}{(\omega^2+1)(\omega^2+1)}} \angle \tan^{-1}\left(\frac{\omega}{2}\right) - \left(\tan^{-1}\left(\frac{\omega}{1}\right) + 180^\circ - \tan^{-1}\left(\frac{\omega}{1}\right)\right)$$

$$= \frac{\sqrt{\omega^2+4}}{(\omega^2+1)} \angle -180^\circ + \tan^{-1}\left(\frac{\omega}{2}\right)$$

$$\omega = 0,\ G(s)\,H(s) = 2\angle -180^\circ = -2$$

$$\omega = \infty,\ G(s)\,H(s) = 0\angle -90^\circ$$



$N = 1$ as Nyquist plot makes one anti-clockwise encirclements of the point ($-1 + j0$) and since $P = 1$.

$$1 = 1 - Z$$

$Z = 0$; Hence, no roots of the characteristic equations are laying on the right hand side of the s-plane, therefore, closed-loop system is stable.

7 Compensators and Controllers

LEVEL 1 Objective Solutions

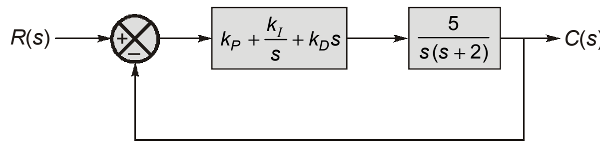
1.	(d)
2.	(b)
3.	(d)
4.	(c)
5.	(a)
6.	(d)
7.	(a)
8.	(a)
9.	(c)
10.	(d)
11.	(d)
12.	(b)
13.	(c)
14.	(d)
15.	(d)
16.	(d)
17.	(c)
18.	(c)
19.	(d)

LEVEL 2 Objective Solutions

20.	(c)
21.	(d)
22.	(c)
23.	(a)
24.	(a)
25.	(c)
26.	(b)
27.	(d)

LEVEL 3 Conventional Solutions

Solution: 1



Let the PID controller be

$$G_c(s) = k_P + \frac{k_I}{s} + k_D s$$

$$T(s) = \frac{C(s)}{R(s)} = \frac{\frac{5}{s(s+2)}\left[\frac{k_P s + k_I + k_D s^2}{s}\right]}{1 + \frac{5[k_P s + k_I + k_D s^2]}{s^2[s+2]}}$$

$$\frac{C(s)}{R(s)} = \frac{5[k_I + k_P s + k_D s^2]}{s^3 + 2s^2 + 5k_D s^2 + 5k_P s + 5k_I}$$

Now desired location of poles are

$$s = -8 \text{ and } s = -3 + 4j$$

$\therefore$ Characteristic equation $= (s + 8)\,(s - (-3 + 4j))\,(s - (-3 - 4j))$

$$= (s + 8)\,(s^2 + 6s + 25)$$

$$= s^3 + 14s^2 + 73s + 200$$

Comparing with C.E $= s^3 + (2 + 5k_D)\,s^2 + 5k_P s + 5k_I$

$\therefore$ $2 + 5k_D = 14$

$\therefore$ $k_D = \frac{12}{5} = 2.4$

$$5k_P = 73$$

$$k_P = 14.6$$

$$5\,k_I = 200$$

$$k_I = 40$$

$$G_c(s) = 14.6 + \frac{40}{s} + 2.4s \qquad \text{(PID controller used)}$$

Solution: 2

Given that

Phase lead provided by lead compensator

$$\phi_m = 45° \qquad \text{...(i)}$$

Gain of the compensator $A = 10$ dB at $\omega = 8$ rad/sec.

General form of transfer function of lead compensator

$$T(s) = \frac{K_c\alpha[1 + sT]}{[1 + \alpha Ts]};\quad 0 < \alpha < 1$$

$$T(j\omega) = \frac{K_c\alpha[1+j\omega T]}{[1+j\alpha\omega T]}$$

$\therefore$ Phase $\phi = \tan^{-1}\omega T - \tan^{-1}\alpha\omega T$

So find the value of maximum phase lead it provide for this

$$\frac{d\phi}{d\omega} = 0$$

$$\frac{d}{d\omega}\left[\tan^{-1}\omega T - \tan^{-1}\alpha\omega T\right] = 0$$

$$\Rightarrow \quad \frac{1}{1+\omega^2T^2}\cdot T - \frac{1}{1+\alpha^2\omega^2T^2}\cdot\alpha T = 0$$

$T \Rightarrow$ can't be zero

$$\frac{1}{1+\omega^2T^2} - \frac{\alpha}{1+\alpha^2\omega^2T^2} = 0$$

$$\frac{1}{1+\omega^2T^2} = \frac{\alpha}{1+\alpha^2\omega^2T^2}$$

$$1+\alpha^2\omega^2T^2 = \alpha\omega^2T^2+\alpha$$

$$1-\alpha = \alpha\omega^2T^2[1-\alpha]$$

$$\alpha\omega^2T^2 = 1$$

$$\boxed{\omega = \frac{1}{\sqrt{\alpha}\,T}} \quad \text{...(i)}$$

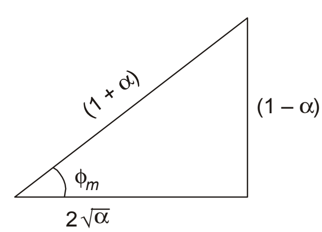
So at $\omega = \dfrac{1}{\sqrt{\alpha}\,T}$ maximum phase lead will occur.

$\therefore$ maximum phase lead

$$\phi_m = \tan^{-1}\frac{1}{\sqrt{\alpha}\,T}\cdot T - \tan^{-1}\frac{\alpha 1\cdot T}{\sqrt{\alpha}\,T}$$

$$\phi_m = \tan^{-1}\frac{1}{\sqrt{\alpha}} - \tan^{-1}\sqrt{\alpha}$$

$$\phi_m = \tan^{-1}\frac{(1-\alpha)}{2\sqrt{\alpha}}$$



or, $$\boxed{\phi_m = \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right)}$$

Given $\phi_m = 45°$

$$45° = \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right)$$

$$\sin 45° = \frac{1-\alpha}{1+\alpha}$$

$$\frac{1}{\sqrt{2}} = \frac{1-\alpha}{1+\alpha}$$

$$1+\alpha = \sqrt{2}-\sqrt{2}\alpha$$

$$(1+\sqrt{2})\,\alpha = (\sqrt{2}-1)$$

$$\alpha = \frac{\sqrt{2}-1}{1+\sqrt{2}}$$

$$\boxed{\alpha = 0.1715}$$

Given that gain at $\omega = 8$ rad/sec is 10 dB

$$A = |T(j\omega)| = \frac{K_c\alpha\sqrt{1+\omega^2T^2}}{\sqrt{1+\alpha^2\omega^2T^2}}$$

$$\text{gain in dB} = \frac{20\log K_c\alpha\sqrt{1+\omega^2T^2}}{\sqrt{1+\alpha^2\omega^2T^2}}$$

$$10 = 20\log \alpha K_c + 10\log(1+\omega^2T^2) - 10\log(1+\alpha^2\omega^2T^2)$$

$$10 = 20\log 0.1715 + 10\log\left(\frac{1+\omega^2T^2}{1+\alpha^2\omega^2T^2}\right) + 20\log K_c \qquad \text{...(ii)}$$

$\because$ $$\omega = \frac{1}{\sqrt{\alpha}T}$$

Given that $\omega = 8$ rad/sec.

$\therefore$ $$T = \frac{1}{\sqrt{\alpha}\omega} = \frac{1}{\sqrt{0.1715}\times 8}$$

$$\boxed{T = 0.3018}$$

Now from equation (ii),

$$10 = 20\log 0.1715 + 10\log\left(\frac{1+\omega^2T^2}{1+\alpha^2\omega^2T^2}\right) + 20\log K_c$$

$$20\log K_c = 17.6581$$

$$\log K_c = (17.6581/20)$$

$$K_c = (10)^{0.882905}$$

$$K_c = 7.6366$$

$$\boxed{K_c = 7.64}$$

$\therefore$ Transfer function will be

$$T(s) = \frac{K_c\alpha(1+Ts)}{(1+\alpha Ts)} = 7.64\times 0.1715\frac{(1+0.3018s)}{(1+0.1715\times 0.3018s)}$$

$$T(s) = 1.31026\frac{(1+0.3018s)}{(1+0.0518s)}$$

Solution : 3

$$G(s) = \frac{20}{s(1+s)(1+2s)}$$

Let controller transfer function be

$$G_c(s) = K_p + sK_d$$

Hence, $$G(s)\,G_c(s) = \frac{20(K_p+sK_d)}{s(1+s)(1+2s)}$$

At $\omega = 1.2$ rad/sec, phase margin = 30°,

$$\text{PM} = 180 + \angle G(j\omega)\,H(j\omega)$$

Hence $$G(j1.2)G_c(j1.2) = 1\angle -(180°) + 30°$$

$$G(j1.2)G_c(j1.2) = \frac{20(K_p + 1.2K_d s)}{j1.2(1+j1.2)(1+j2.4)}$$

Hence,
$$-150° = -90° - \tan^{-1}(1.2) - \tan^{-1}(2.4) + \tan^{-1}\left(\frac{1.2K_d}{K_p}\right)$$

$$-150° = -90° - 50.194° - 67.38° + \tan^{-1}\left(\frac{1.2K_d}{K_p}\right)$$

$$\frac{1.2K_d}{K_p} = 1.574 \quad \text{...(i)}$$

$$|G(j1.2)G_c(j1.2)| = \left|\frac{20(K_p + j1.2K_d)}{j1.2(1+j1.2)(1+j2.4)}\right| = 1$$

Therefore,
$$\frac{20\sqrt{K_p^2 + (1.2K_d)^2}}{1.2 \times 1.562 \times 2.6} = 1$$

$$\sqrt{K_p^2 + (1.2K_d)^2} = 0.2436 \quad \text{...(ii)}$$

$$K_p\sqrt{1+\left(\frac{1.2K_d}{K_p}\right)^2} = 0.243$$

From (i) and (ii), we get

$$K_p\sqrt{1+1.574^2} = 0.2436$$

or
$$K_p = 0.1306$$

Putting this value in (i), we get

$$K_d = 0.171$$

Controller transfer function,

$$G_c(s) = K_p + K_d s = 0.1306 + 0.171s = 0.1306(1 + 1.309s)$$

Hence,
$$G(s)\,G_c(s) = \frac{20 \times 0.1306(1+1.309s)}{s(1+s)\,(1+2s)}$$

$$= \frac{(2.162)\,(1+1.309s)}{s(1+s)\,(1+2s)}$$

We find that at $G(j12)\ G_c(j1.2) = 1\angle -150°$

Hence, phase margin $= 180° - 150° = 30°$

Hence, transfer function PD controller $= 0.1306(1+1.309s)$

Solution : 4

Let the controller transfer function be $G_c(s) = K_p + \frac{K_i}{s}$.

Hence, $$G(s)\,H(s)\,G_c(s) = \frac{4(K_i + sK_p)}{s(s+1)(s+2)}$$

System is to have phase margin of 50° at 1.7 rad/sec.

Hence, $$G(j1.7)\,H(j1.7)\,G_c(j1.7) = 1\angle -(180° - 50°) = 1\angle -130°$$

$$G(j1.7)\,H(j1.7)\,G_c(j1.7) = \frac{4(K_i + j1.7K_p)}{j(1.7)(1+j1.7)(2+j1.7)}$$

Hence, $$\angle G(j1.7)\,H(j1.7)\,G_c(j1.7) = -90° - \tan^{-1}(1.7) - \tan^{-1}\left(\frac{1.7}{2}\right) + \tan^{-1}\left(\frac{1.7K_p}{K_i}\right)$$

From this, we get $\frac{1.7K_p}{K_i} = \tan 59.898°$

Hence, $$\frac{1.7K_p}{K_i} = 1.7249 \qquad \text{...(i)}$$

Also, $$\left|G(j1.7)\,H(j1.7)\,G_c(j1.7)\right| = \left|\frac{4(K_i + j1.7K_p)}{j1.7(1+j1.7)\,(2+j1.7)}\right| = 1$$

or $$\frac{4\sqrt{K_i^2 + (1.7K_p)^2}}{1.7 \times 1.972 \times 2.624} = 1$$

or $$\sqrt{K_i^2 + (1.7K_p)^2} = 2.199$$

or $$K_i\sqrt{1 + \left(\frac{1.7K_p}{K_i}\right)^2} = 2.199$$

From (i), we have $$\frac{1.7K_p}{K_i} = 1.7249$$

Hence $$K_i\sqrt{1. + 1.7249^2} = 2.199$$

So, $$K_i = 1.1029$$

Putting this value in (i), we get

$$K_p = 1.119$$

Hence, $$G_c(s) = 1.119 + \frac{1.1029}{s}$$

Solution : 5

$$G(s)\,H(s) = \frac{1}{s(s+1)}$$

Let the $$G_c(s) = K_p + sK_d$$

Hence, $$G(s)\,H(s)\,G_c(s) = \frac{(K_d + sK_d)}{s(s+1)}$$

At ω = 2 rad/sec, phase margin = 40°, hence

$$G(j2)\,H(j2)\,G_c(j2) = 1\angle(180° - 40°)$$

$$G(j2)\,H(j2)\,G_c(j2) = \frac{(K_d + j2K_d)}{j(1+j2)}$$

Hence, $$-140° = -90° - \tan^{-1}2 + \tan^{-1}\left(\frac{2K_d}{K_p}\right)$$

or $$-140° = -90° - 63.434° + \tan^{-1}\left(\frac{2K_d}{K_p}\right)$$

So, $$\tan^{-1}\left(\frac{2K_d}{K_p}\right) = 13.434$$

or $$\frac{2K_d}{K_p} = 0.2388 \qquad \text{...(i)}$$

$$|G(j2)\,H(j2)\,G_c(j2)| = \left|\frac{(K_d + j2K_d)}{j(1+j2)}\right| = 1$$

Therefore, $$\frac{\sqrt{K_p^2 + (2K_d)^2}}{2 \times 2.236} = 1$$

or $$\sqrt{K_p^2 + (2K_d)^2} = 4.472$$

$$K_p\sqrt{1 + \left(\frac{2K_d}{K_p}\right)^2} = 4.472 \qquad \text{...(ii)}$$

From (i) and (ii), we get $$K_p\sqrt{1 + 0.2388^2} = 4.472$$

$$K_p = 4.349$$

Putting this value in (1), we get,

$$K_d = 0.519$$

Controller transfer function $$G_c(s) = K_p + K_d s$$
$$= 4.349 + 0.5198$$

Solution : 6

Let the transfer function of the controller be

$$G_c(s) = K_p + sK_d$$

Then, we have $$G(s)H(s)G_c(s) = \frac{10(K_p + sK_d)}{s(s+3)}$$

Hence, the characteristics equation is given by,

$$1+\frac{10(K_p+sK_d)}{s(s+3)} = 0 \qquad ...(i)$$

$s = -1.5\pm j\sqrt{5} = 2.692\angle 123.855°$ is dominant pole of the system. Hence, it will satisfy (i).

Therefore,

$$1+\frac{10(K_p+2.692K_d\angle 123.855)}{(-1.5+j2.236)\,(-1.5+j2.236+3)} = 0$$

$$1+\frac{10(K_p+2.692K_d\angle 123.855)}{2.692\angle 123.855\times 2.692\times\angle 56.145} = 0$$

$1 + (1.3799\angle -180°)\,(K_p + 2.692K_d\angle 123.855°) = 0$

$1 + 1.3799K_p\angle -180° + 3.714K_d\angle -56.145° = 0$

$1 - 1.3799K_p + 2.069K_d - j3.084K_d = 0$

Equating real and imaginary parts to zero, we get

$$1 - 1.3799K_p + 2.069K_d = 0$$

$$j3.084K_d = 0$$

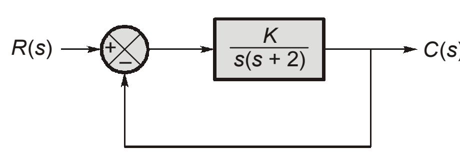
From these equation, we get $K_d = 0$

$$K_p = 0.7246$$

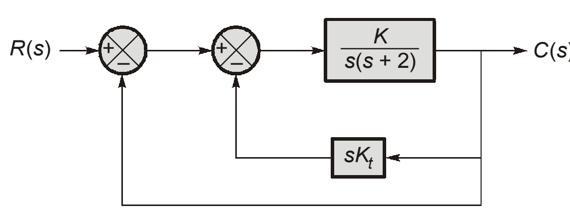
Hence, $G_c(s) = 0.7246$

Solution : 7

The block diagram of uncompensated system is drawn in figure.



With tachometer feedback (feedback compensation) the block diagram of figure 1 is redrawn as shown in figure 2.



The overall transfer function of the compensated system is

$$\frac{C(s)}{R(s)} = \frac{K}{s^2+(2+KK_t)s+K}$$

Since $M_p = 25\%$

$$0.25 = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$$

$$-1.38 = -\frac{\pi\xi}{\sqrt{1-\xi^2}} \times 1 \qquad \Rightarrow \xi = 0.4$$

$$t_p = 1 \text{ sec}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} \Rightarrow \omega_n\sqrt{1-\xi^2} = \pi$$

$$\omega_n = \frac{\pi}{\sqrt{1-\xi^2}} = \frac{\pi}{\sqrt{1-(0.4)^2}} = 3.42 \text{ rad/sec}$$

The characteristic equation for the compensated system is

$$s^2 + (2 + KK_T)\, s + K = 0$$

From the characteristic equation it is noted that

$$\omega_n = \sqrt{k} \text{ and } 2\xi\omega_n = (2 + KK_t)$$

$\therefore$ $$3.42 = \sqrt{k}$$

$$k = (3.42)^2 = 11.7$$

$$2\xi\omega_n = 2 + KK_t$$

$$2 \times 0.4 \times 3.42 = 2 + 11.7K_t$$

$$K_t = 0.065$$

$\therefore$ $$K = 11.7$$

and $$K_t = 0.065$$

■■■■

8 State Space Analysis

LEVEL 1 Objective Solutions

1.	(–1)
2.	(a)
3.	(b)
4.	(c)
5.	(2.5)
6.	(a)
7.	(b)
8.	(a)
9.	(a)
10.	(b)
11.	(a)
12.	(b)
13.	(a)
14.	(c)
15.	(d)
16.	(d)
17.	(c)
18.	(b)
19.	(d)

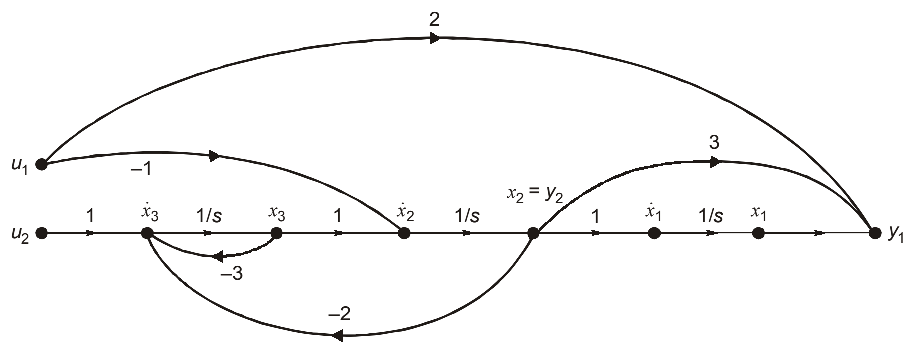
LEVEL 2 Objective Solutions

20.	(b)
21.	(a)
22.	(a)
23.	(c)
24.	(a)
25.	(b)
26.	(d)
27.	(a)
28.	(b)
29.	(a)
30.	(c)
31.	(a)
32.	(c)
33.	(d)
34.	(d)
35.	(a)
36.	(d)
37.	(b)
38.	(a)
39.	(b)
40.	(b)

LEVEL 3 Conventional Solutions

Solution: 1

Signal flow graph,



From the given state space equations,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Here system matrix, $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}$$

Characteristic roots of system

$$|sI - A| = 0$$

$$\Rightarrow \quad \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \right| = 0$$

$$\Rightarrow \quad \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 2 & s+3 \end{vmatrix} = 0$$

$$\Rightarrow \quad s(s(s+3)+2) = 0$$

$$\Rightarrow \quad s(s^2 + 3s + 2) = 0$$

$$\Rightarrow \quad s^3 + 3s^2 + 2s = 0$$

$$\Rightarrow \quad s(s+1)(s+2) = 0$$

$$s = 0, -1, -2$$

Characteristic roots of system are 0, –1, –2.

Solution: 2

$$\dot{X} = AX$$

Take Laplace transform

$$\Rightarrow \quad SX(s) - x(0) = AX(s) \qquad \text{...(1)}$$

For, $$x(0) = \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} e^{-2t} \\ -\frac{3}{2}e^{-2t} \end{bmatrix}$$

$\therefore$ $$X(s) = \begin{bmatrix} \frac{1}{(s+2)} \\ \frac{-3}{2(s+2)} \end{bmatrix}$$

From (1)

$$\begin{bmatrix} \frac{s}{(s+2)} \\ \frac{-3s}{2(s+2)} \end{bmatrix} - \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} = A\begin{bmatrix} \frac{1}{(s+2)} \\ \frac{-3}{2(s+2)} \end{bmatrix} \quad \text{...(2)}$$

For, $$x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$$

$\therefore$ $$X(s) = \begin{bmatrix} \frac{1}{s+1} \\ \frac{-1}{s+1} \end{bmatrix}$$

From (1)

$$\begin{bmatrix} \frac{s}{s+1} \\ \frac{-s}{s+1} \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = A\begin{bmatrix} \frac{1}{s+1} \\ \frac{-1}{s+1} \end{bmatrix}$$

$\Rightarrow$ $$\begin{bmatrix} \frac{s}{s+1} - 1 \\ \frac{-s}{s+1} + 1 \end{bmatrix} = A\begin{bmatrix} \frac{1}{s+1} \\ \frac{-1}{s+1} \end{bmatrix}$$

$\Rightarrow$ $$\begin{bmatrix} \frac{-1}{s+1} \\ \frac{1}{s+1} \end{bmatrix} = A\begin{bmatrix} \frac{1}{s+1} \\ \frac{-1}{s+1} \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} -1 \\ 1 \end{bmatrix} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad ...(3)$$

From (2)

$$\begin{bmatrix} \frac{s}{s+2} - 1 \\ \frac{-3s}{2(s+2)} + \frac{3}{2} \end{bmatrix} = A\begin{bmatrix} \frac{1}{s+2} \\ \frac{-3}{2(s+2)} \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} \frac{-2}{s+2} \\ \frac{3}{s+2} \end{bmatrix} = A\begin{bmatrix} \frac{1}{s+2} \\ \frac{-3}{2(s+2)} \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} -2 \\ 3 \end{bmatrix} = A\begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} \qquad ...(4)$$

Let, $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\therefore$ From (3)

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} a-b \\ c-d \end{bmatrix}$$

$\therefore$ $$a - b = -1 \qquad ...(5)$$

$$c - d = 1 \qquad ...(6)$$

From (4)

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix}$$

$$a - \frac{3b}{2} = -2 \qquad ...(7)$$

$$c - \frac{3d}{2} = 3 \qquad ...(8)$$

Equation (5) - (7)

$$\frac{b}{2} = 1$$

$\Rightarrow$ $$b = 2$$

$\therefore$ $$a = 1$$

Equation (6) - (8)

$$\frac{d}{2} = -2$$

$\Rightarrow$
$$d = -4$$
$$c = -3$$

$\therefore$
$$A = \begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix}$$

Now, resolvent matrix
$$\phi(s) = [sI - A]^{-1}$$

$\Rightarrow$
$$\phi(s) = \left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix}\right]^{-1} = \begin{bmatrix} s-1 & -2 \\ 3 & s+4 \end{bmatrix}^{-1}$$

$$= \frac{1}{(s-1)(s+4)+6}\begin{bmatrix} s+4 & 2 \\ -3 & s-1 \end{bmatrix}$$

$$= \frac{1}{s^2+4s-s-4+6}\begin{bmatrix} s+4 & 2 \\ -3 & s-1 \end{bmatrix}$$

$$= \frac{1}{s^2+3s+2}\begin{bmatrix} s+4 & 2 \\ -3 & s-1 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+4 & 2 \\ -3 & s-1 \end{bmatrix}$$

$$= \left[\frac{1}{s+1} - \frac{1}{s+2}\right]\begin{bmatrix} s+4 & 2 \\ -3 & s-1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+4}{s+1} - \frac{s+4}{s+2} & \frac{2}{s+1} - \frac{2}{s+2} \\ \frac{-3}{s+1} + \frac{3}{s+2} & \frac{s-1}{s+1} - \frac{s-1}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} 1+\frac{3}{s+1} - 1 - \frac{2}{s+2} & \frac{2}{s+1} - \frac{2}{s+2} \\ -\frac{3}{s+1} + \frac{3}{s+2} & 1 - \frac{2}{s+1} - 1 + \frac{3}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{s+1} - \frac{2}{s+2} & \frac{2}{s+1} - \frac{2}{s+2} \\ \frac{-3}{s+1} + \frac{3}{s+2} & \frac{-2}{s+1} + \frac{3}{s+2} \end{bmatrix}$$

Taking Laplace inverse we can write the state transition matrix as follows:

$$\phi(t) = \begin{bmatrix} 3e^{-t} - 2e^{-2t} & 2e^{-t} - 2e^{-2t} \\ -3e^{-t} + 3e^{-2t} & -2e^{-t} + 3e^{-2t} \end{bmatrix}$$

Solution: 3

Transfer function

$$\frac{Y(s)}{U(s)} = \frac{s^2+3s+4}{s^3+2s^2+3s+2}$$

$$\frac{Y(s)}{X_1(s)}\cdot\frac{X_1(s)}{U(s)} = \frac{1}{(s^3+2s^2+3s+2)}\cdot(s^2+3s+4)$$

Case (i)

$$\frac{X_1(s)}{U(s)} = \frac{1}{s^3+2s^2+3s+2}$$

$$\Rightarrow \quad X_1(s)\,(s^3+2s^2+3s+2) = U(s)$$

$$\Rightarrow \quad \frac{d^3x_1(t)}{dt^3}+2\frac{d^2x_1(t)}{dt^2}+3\frac{dx_1(t)}{dt}+2x_1(t) = u$$

Let,

$$\frac{dx_1(t)}{dt} = x_2$$

$$\frac{d^2x_1(t)}{dt^2} = \dot{x}_2 = x_3$$

$$\frac{d^3x_1(t)}{dt^3} = \dot{x}_3 = -2x_3-3x_2-2x_1+4$$

Output equation

$$\frac{Y(s)}{X_1(s)} = (s^2+3s+4)$$

$$\Rightarrow \quad Y(s) = (s^2+3s+4)\,X_1(s)$$

$$y(t) = \frac{d^2x_1(t)}{dt^2}+3\frac{dx_1(t)}{dt}+4x_1(t) = 4x_1+3x_2+x_3$$

State model of system in phase variable form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & -3 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}4 & 3 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

For observability

Observability matrix; $$O_M = \begin{bmatrix}C\\ CA\\ CA^2\end{bmatrix}$$

$$CA = \begin{bmatrix}4 & 3 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 1\\ 0 & 0 & 1\\ -2 & -3 & -2\end{bmatrix} = \begin{bmatrix}-2 & 1 & 1\end{bmatrix}$$

$$CA^2 = CA\,(A) = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -2 \end{bmatrix}$$

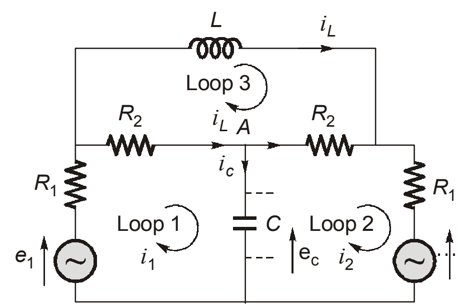
$$= \begin{bmatrix} -2 & -5 & -1 \end{bmatrix}$$

therefore; OM $$= \begin{bmatrix} 4 & 3 & 1 \\ -2 & 1 & 1 \\ -2 & -5 & -1 \end{bmatrix}$$

O_M matrix is non-singular matrix hence the system is observable.
The smallest time constant of system is 1 sec i.e. τ = 1 sec.

Solution: 4

(i) Let us choose i_L and e_C to be the state variables.



KCL at node '*A*',

$$-i_1 + i_C + i_2 = 0$$

$\Rightarrow$ $$i_C = i_1 - i_2$$

Applying KVL in Loop 1,

$$R_1 i_1 + R_2(i_1 - i_L) = e_1 - e_C$$

or $$(R_1 + R_2)i_1 - R_2 i_L = e_1 - e_C \quad \text{...(i)}$$

Applying KVL in Loop 2,

$$R_2(i_2 - i_L) + R_1 i_2 = e_C - e_2$$

or $$i_2(R_1 + R_2) - R_2 i_L = e_C - e_2 \quad \text{...(ii)}$$

On solving equation (i) and (ii) we get,

$$i_1 = \frac{e_1 - e_c + R_2 i_L}{R_1 + R_2}$$

and $$i_2 = \frac{e_c - e_2 + R_2 i_L}{R_1 + R_2}$$

Now applying KVL in Loop 3,

$$L\frac{di_L}{dt} + R_2(i_L - i_2) + R_2(i_L - i_1) = 0$$

or $$R_2 i_1 + R_2 i_2 = L\frac{di_L}{dt} + 2R_2 i_L \quad \text{...(iii)}$$

Putting value of i_1 and i_2 in the equation (iii),

$$\Rightarrow \quad R_2\left\{\frac{e_1 - e_c + R_2 i_L}{R_1 + R_2}\right\} + R_2\left\{\frac{e_C - e_2 + R_2 i_L}{R_1 + R_2}\right\} = L\frac{di_L}{dt} + 2R_2 i_L$$

$$\Rightarrow \quad \frac{R_2}{R_1 + R_2}\{e_1 - e_2 + 2R_2 i_L\} = L\dot{i}_L + 2R_2 i_L \qquad \left[\because \frac{di_L}{dt} = \dot{i}_L\right]$$

$$\Rightarrow \quad L\dot{i}_L = -2R_2 i_L\left(1 - \frac{R_2}{R_1 + R_2}\right) + \frac{R_2}{R_1 + R_2}(e_1 - e_2)$$

$$\Rightarrow \quad L\dot{i}_L = \left(\frac{-2R_1R_2}{R_1 + R_2}\right)i_L + \frac{R_2}{R_1 + R_2}(e_1 - e_2)$$

$$\Rightarrow \qquad \dot{i}_L = \frac{-2R_1R_2}{(R_1+R_2)L}i_L + \frac{R_2}{(R_1+R_2)L}(e_1-e_2) \qquad \text{...(iv)}$$

Since,
$$i_C = C\frac{de_c}{dt} = C\dot{e}_c = i_1 - i_2$$

$$\therefore \qquad C\dot{e}_c = -i_2 + i_1 \qquad \text{...(v)}$$

Now, putting values of i_1 and i_2 in equation (v), we get,

$$\Rightarrow \qquad C\dot{e}_c = -\left(\frac{e_C - e_2 + R_2 i_L}{R_1+R_2}\right) + \left(\frac{e_1 - e_C + R_2 i_L}{R_1+R_2}\right)$$

$$\Rightarrow \qquad C\dot{e}_c = \left(\frac{e_2 + e_1 - 2e_C}{R_1+R_2}\right)$$

$$\therefore \qquad \dot{e}_c = \frac{2}{(R_1+R_2)C}e_c + \frac{1}{(R_1+R_2)C}(e_1+e_2) \qquad \text{... (vi)}$$

Writing the equation (iv) and (vi) in the required state variable form as,

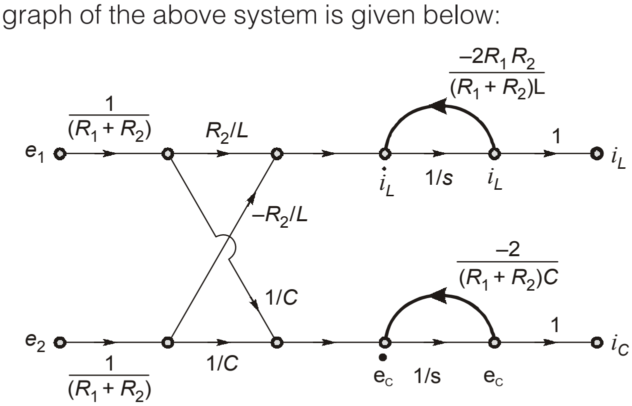
$$\begin{bmatrix} \dot{i}_L \\ \dot{e}_c \end{bmatrix} = \begin{bmatrix} \dfrac{-2R_1R_2}{(R_1+R_2)L} & 0 \\ 0 & \dfrac{-2}{(R_1+R_2)C} \end{bmatrix} \begin{bmatrix} i_L \\ e_c \end{bmatrix} + \begin{bmatrix} \dfrac{R_2}{(R_1+R_2)L} & -\dfrac{R_2}{(R_1+R_2)L} \\ \dfrac{+1}{(R_1+R_2)C} & \dfrac{+1}{(R_1+R_2)C} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Comparing with $\dot{X} = AX + BU$ we have,

$$A = \begin{bmatrix} \dfrac{-2R_1R_2}{(R_1+R_2)L} & 0 \\ 0 & \dfrac{-2}{(R_1+R_2)C} \end{bmatrix}$$

and
$$B = \begin{bmatrix} \dfrac{R_2}{(R_1+R_2)L} & \dfrac{-R_2}{(R_1+R_2)L} \\ \dfrac{1}{(R_1+R_2)C} & \dfrac{1}{(R_1+R_2)C} \end{bmatrix}$$

(ii) State model flow graph of the above system is given below:



(iii) Advantages of state space analysis over the conventional differential equation methods:

1. The state variable analysis gives the complete information about the internal states of the system at any given point of time.

2. It takes initial conditions into consideration.
3. It is applicable for multiple-input-multiple-output system.
4. It is applicable for both LTI and time-varying systems.
5. The controllability and observability can be determined easily.
6. It employs the use of vector matrix notation.
7. Representation of higher order system thus become very simple through this technique.

Solution : 5

Let $x_1(t)$, $x_2(t)$ and $x_3(t)$ are the state variables.

Let,
$$x_1(t) = y(t) \quad \text{...(i)}$$
$$x_2(t) = \dot{y}(t)$$
$$x_3(t) = \ddot{y}(t)$$

Differentiating the above equations

$$\dot{x}_1(t) = \dot{y}(t) = x_2(t) \quad \text{...(ii)}$$
$$\dot{x}_2(t) = \ddot{y}(t) = x_3(t) \quad \text{...(iii)}$$
$$\dot{x}_3(t) = \dddot{y}(t) = 8u(t) - 5\ddot{y}(t) - 10\dot{y}(t) - 10y(t)$$
$$\dot{x}_3(t) = 8u(t) - 5x_3(t) - 10x_2(t) - 10x_1(t) \quad \text{...(iv)}$$

From (i), (ii), (iv)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -10 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} u(t)$$

$$y(t) = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution : 6

$$I = i_c + i_L$$
$$I = C\frac{dv_c}{dt} + i_L$$
$$\frac{dv_c}{dt} = \frac{I}{C} - \frac{i_L}{C} \quad \text{...}(i)$$
$$v_c = L\frac{di_L}{dt} + i_L R$$
$$\frac{di_L}{dt} = \frac{V_c}{L} - \frac{R}{L}i_L \quad \text{...(ii)}$$

Output
$$v_0(t) = Ri_L \quad \text{...(iii)}$$

From (i), (ii) and (iii)

$$\begin{bmatrix} \dfrac{dv_c}{dt} \\ \dfrac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{1}{C} \\ \dfrac{1}{L} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} \dfrac{1}{C} \\ 0 \end{bmatrix} I$$

$$v_0 = [O \quad R] \begin{bmatrix} v_c \\ i_L \end{bmatrix}$$

Solution : 7

The given state variable equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

(i) Characteristic equation

$$|sI - A| = 0$$

$$\Rightarrow \quad \begin{vmatrix} s-1 & -2 & -1 \\ 0 & s-1 & -3 \\ -1 & -1 & s-1 \end{vmatrix} = 0$$

$$\Rightarrow (s-1)\,[(s-1)^2 - 3] - 1[6 + (s-1)] = 0$$

$$\Rightarrow (s-1)\,(s^2 + 1 - 2s - 3) - (s+5) = 0$$

$$\Rightarrow \quad (s-1)\,(s^2 - 2s - 2) - (s+5) = 0$$

$$\Rightarrow \quad s^3 - 3s^2 - s - 3 = 0$$

(ii) Controllability matrix

$$M = [B\ AB\ A^2B \ldots\ldots A^{(n-1)}\ B]_{n \times n}$$

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{3\times 3} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}_{3\times 3}$$

$$A^2B = \begin{bmatrix} 2 & 5 & 8 \\ 3 & 4 & 6 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \\ 7 \end{bmatrix}$$

$\therefore$ Controlling matrix $\quad M = [B\ AM\ A^2B]$

$$= \begin{bmatrix} 1 & 2 & 10 \\ 0 & 3 & 9 \\ 1 & 2 & 7 \end{bmatrix}$$

(iii) For controllability, $(M) \neq 0$. i.e., the rank of M should be 'n'. If rank is $(n-k)$, implies 'k' number of state variable are not controllable.

$$|M| = \begin{bmatrix} 1 & 2 & 10 \\ 0 & 3 & 9 \\ 1 & 2 & 7 \end{bmatrix}$$

$$= 1(21-18) - 2(0-9) - 10(0-3)$$

$$= 3 + 18 - 30 = -9$$

The system is controllable.

Solution : 8

The given transfer function $G(s) = \dfrac{3s^2+5s+7}{s^4+5s^3+18s^2+29s+35}$

(i) Controller canonical form

Let $$\frac{Y(s)}{U(s)} = \frac{3s^2+5s+7}{s^4+5s^3+18s^2+29s+35}$$

Decomposed transfer function

$$\xrightarrow{U(s)} \frac{1}{s^4+5s^3+18s^2+29s+35} \xrightarrow{X(s)} 3s^2+5s+7 \xrightarrow{Y(s)}$$

$\therefore$ $$\frac{X(s)}{U(s)} = \frac{1}{s^4+5s^3+18s^2+29s+35}$$

$s^4X(s) + 5s^3X(s) + 18s^2X(s) + 29sX(s) + 35X(s) = U(s)$

By taking inverse Laplace transform on both sides,

$$\frac{d^4x}{dt^4} + 5\frac{d^3x}{dt^3} + \frac{18d^2x}{dt^2} + \frac{29dx}{dt} + 35x = u$$

Let the state variable are X_1, X_2, X_3 and X_4.

$$X_1 = x$$

$$X_2 = \frac{dx}{dt} \qquad \Rightarrow \dot{X}_1 = X_2$$

$$X_3 = \frac{d^2x}{dt} \Rightarrow \frac{dx_2}{dt} \Rightarrow \dot{X}_1 = X_2$$

$$X_4 = \frac{d^3x}{dt^3} \Rightarrow \frac{dx_3}{dt} \Rightarrow \dot{X}_3 = X_4$$

From the above differential equation

$$\dot{X}_4 = u - 5X_4 - 18X_3 - 29X_2 - 35X_1$$

$$\frac{Y(s)}{X(s)} = 3s^2 + 5s + 7$$

$\Rightarrow$ $$Y(s) = 3s^2X(s) + 5sX(s) + 7X(s)$$

By taking inverse Laplace transform on both sides

$$Y(s) = 3\frac{d^2X(t)}{dt^2} + 5\frac{dX(t)}{dt} + 7X(t) = 3X_3 + 5X_2 + 7X_1$$

$\therefore$ The state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -35 & -29 & -18 & -5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} U$$

The output equation is

$$Y = [7 \ \ 5 \ \ 3 \ \ 0] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

(ii) Observable canonical form:

$$A_{OCF} = \text{Transpose of } A_{CCF}$$

i.e.

$$A_{OCF} = \begin{bmatrix} 0 & 0 & 0 & -35 \\ 1 & 0 & 0 & -29 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

$$B_{OCF} = \text{Transpose of } C_{CCF}$$

$$= \begin{bmatrix} 7 \\ 5 \\ 3 \\ 0 \end{bmatrix}$$

$$C_{OCF} = \text{Transpose of } B_{CCF} = [0 \ \ 0 \ \ 0 \ \ 1]$$

$\therefore$ The state equation is,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -35 \\ 1 & 0 & 0 & -29 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \\ 3 \\ 0 \end{bmatrix} [U]$$

$$Y = [0 \ \ 0 \ \ 0 \ \ 1] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

Solution : 9

Time response is given by

$$\mathbf{x}(t) = \phi(t)\,x(0)$$

$$\phi(t) = L^{-1}[sI - A]^{-1}$$

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{s}{s^2+2} & \dfrac{1}{s^2+2} \\ \dfrac{-2}{s^2+2} & \dfrac{s}{s^2+2} \end{bmatrix}$$

$$\phi(s) = (sI - A)^{-1} = \begin{bmatrix} \dfrac{s}{s^2+2} & \dfrac{1}{s^2+2} \\ \dfrac{-2}{s^2+2} & \dfrac{s}{s^2+2} \end{bmatrix}$$

$\therefore$ The state transition matrix $\phi(t)$ is

$$\phi(t) = L^{-1}\phi(s) = L^{-1}\begin{bmatrix} \dfrac{s}{s^2+2} & \dfrac{1}{s^2+2} \\ \dfrac{-2}{s^2+2} & \dfrac{s}{s^2+2} \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} \cos\sqrt{2}t & \dfrac{1}{\sqrt{2}}\sin\sqrt{2}t \\ -\dfrac{2}{\sqrt{2}}\sin\sqrt{2}t & \cos\sqrt{2}t \end{bmatrix} = \begin{bmatrix} \cos\sqrt{2}t & \dfrac{1}{\sqrt{2}}\sin\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t & \cos\sqrt{2}t \end{bmatrix}$$

$$x(t) = \phi(t)x(0)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cos\sqrt{2}t & \dfrac{1}{\sqrt{2}}\sin\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t & \cos\sqrt{2}t \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_1(t) = \cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t$$

$$x_2(t) = -\sqrt{2}\sin\sqrt{2}t + \cos\sqrt{2}t$$

$$y = x_1 - x_2$$

$$y = \frac{3}{\sqrt{2}}\sin\sqrt{2}\,t$$

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