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Leading Institute for ESE, GATE & PSUs

Detailed Solutions

ESE-2026 Mains Test Series

E & T Engineering Test No : 1

Section A : Network Theory

Q.1 (a) Solution:

Source is a basic network element which supplies energy to the networks. There are two classes of sources, namely,

Independent sources and dependent sources

Independent sources:

Output characteristics of an independent source are not dependent on any network variable such as current or voltage. There are two types of independent sources: Independent voltage source and independent current source.

An independent voltage source and current source are two terminal network element which produces a specified voltage and current respectively. The value of voltage and current at any instant is independent of value of direction of current through voltage source and voltage across current source at any instant of time. The terminal voltage in case of voltage source and output current in case of current source may be a constant (DC source) or may be a function of time (AC source).

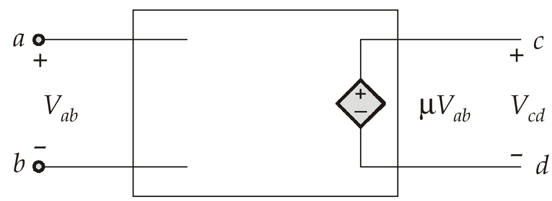
Dependent Sources:

If the voltage or current of a source depends upon some other voltage or current, it is called as dependent or controlled source.

The dependent sources are of four kinds, depending on whether the control variable is voltage or current and the controlled source is a voltage source or current source.

Voltage-Controlled Voltage Source (VCVS):

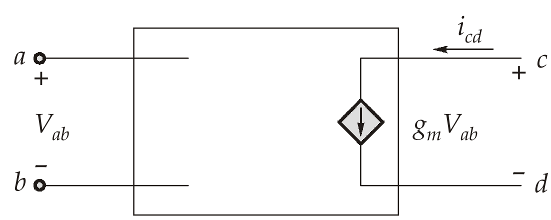
A voltage-controlled voltage source is a four-terminal network component that establishes a voltage between two points *c* and *d* in the circuit that is proportional to a voltage V_{ab} between two points *a* and *b* i.e.,



$$V_{cd} = \mu V_{ab}$$

Voltage-Controlled Current Source (VCCS):

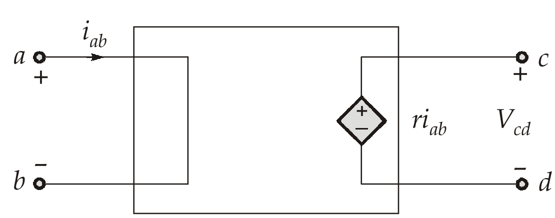
A voltage-controlled current source is a four terminal network component that establishes a current i_{cd} in a branch of the circuit that is proportional to the voltage V_{ab} between two points *a* and *b* i.e.,



$$i_{cd} = g_m V_{ab}$$

Current-Controlled Voltage Source (CCVS):

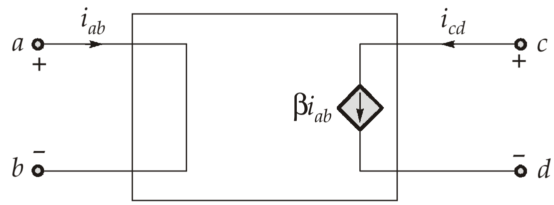
A current-controlled voltage source is a four terminal network component that establishes a voltage V_{cd} between two points *c* and *d* in the circuit that is proportional to the current i_{ab} in some branch of the circuit.



$$V_{cd} = ri_{ab}$$

Current-Controlled Current Source (CCCS):

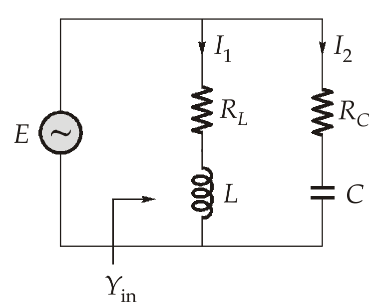
A current-controlled current source is a four terminal network component that establishes a current i_{cd} in one branch of a circuit that is proportional to the current i_{ab} in some branch of the network.

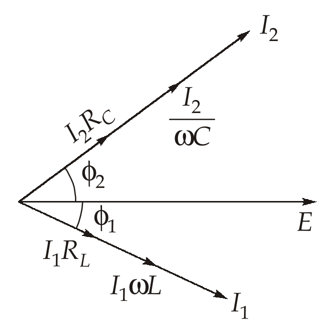


$$i_{cd} = \beta i_{ab}$$

Q.1 (b) Solution:

For a $R_L L$ network, the current I_1 lags the voltage E across it by an angle $\phi_1 = \tan^{-1}(\omega R_L L)$ and for an RC network, the current I_2 leads the voltage across it by an angle $\phi_2 = \tan^{-1}(1/\omega R_C C)$. Thus, the phasor diagram can be drawn as below:





Phasor diagram

The input admittance for the network is given by

$$Y_{in} = \frac{1}{R_L + j\omega L} + \frac{1}{R_C - \frac{j}{\omega C}}$$

$$Y_{in} = \frac{1}{R_L + j\omega L} + \frac{\omega C}{R_C \omega C - j}$$

$$Y_{in} = \frac{R_L - j\omega L}{R_L^2 + (\omega L)^2} + \frac{\omega C(R_C \omega C + j)}{(R_C \omega C)^2 + 1}$$

On separating real and imaginary parts, we get

$$Y_{in} = \left[\frac{R_L}{R_L^2 + (\omega L)^2} + \frac{R_C(\omega C)^2}{1 + (R_C \omega C)^2}\right] + j\left[\frac{-\omega L}{R_L^2 + (\omega L)^2} + \frac{\omega C}{1 + (R_C \omega C)^2}\right]$$

For resonance, Y_{in} should be real, hence its imaginary part should be zero. Thus,

$$\frac{\omega L}{R_L^2 + (\omega L)^2} = \frac{\omega C}{1 + (R_C \omega C)^2}$$

$$\frac{L}{C} = \frac{R_L^2 + (\omega L)^2}{1 + (R_C \omega C)^2}$$

Hence, for the circuit to be in resonance

$$\frac{L}{C} = \frac{R_L^2 + (\omega L)^2}{1 + (R_C \omega C)^2}$$

or, $$L(1 + (R_C\,\omega C)^2) = CR_L^2 + (\omega L)^2 C$$

$$L + R_C^2 C^2 \omega^2 L = CR_L^2 + CL^2\omega^2$$

$$\omega^2 = \frac{CR_L^2 - L}{LR_C^2 C^2 - CL^2}$$

On rearranging, $$\omega^2 = \frac{R_L^2 - L/C}{LR_C^2 C - L^2}$$

or $$\omega^2 = \frac{1}{CL}\left[\frac{R_L^2 - L/C}{R_C^2 - L/C}\right]$$

$$\omega = \frac{1}{\sqrt{LC}}\sqrt{\frac{R_L^2 - L/C}{R_C^2 - L/C}}\ \text{rad/sec}$$

Q.1 (c) Solution:

With iron core I = 4 A, pf = 0.5, V = 200 V

Without iron core I = 5 A, pf = 0.8, V = 40 V

When the iron core is removed,

$$Z = \frac{V}{I} = \frac{40}{5} = 8\ \Omega$$

We have, $$pf = \frac{R}{Z}$$

$$0.8 \times Z = R$$

$$R = 0.8 \times 8$$

$$R = 6.4\ \Omega$$ ....(Value of pure resistance)

$$X_L = \sqrt{Z^2 - R^2}$$

$$X_L = \sqrt{8^2 - 6.4^2} = 4.8\ \Omega$$

$$2\pi fL = 4.8$$

$$L = \frac{4.8}{2\pi \times 50} = 0.0153 \text{ H}$$

With iron core:

$$Z = \frac{200}{4} = 50\ \Omega$$

$$\text{pf} = \frac{R_T}{Z}$$

$0.5 \times 50 = R_T$, where R_T is the resistance of coil with iron core

$$R_T = 25\ \Omega$$

$$X_L = \sqrt{Z^2 - R_T^2} = \sqrt{50^2 - 25^2}$$

$$X_L = 43.30\ \Omega$$

$$L = \frac{43.30}{2\pi \times 50}$$

$$L = 0.1378 \text{ H}$$

Now, iron loss, P_i = [Total active power loss (with iron core)] – [loss due to real resistance]

$$= VI\cos\phi - I^2R$$

$$= 200 \times 4 \times 0.5 - 4^2 \times \frac{64}{10}$$

$$= 297.6 \text{ Watt}$$

Q.1 (d) Solution:

On applying KVL to Mesh 1, we get

$$30\angle 0° - 5I_1 - j5(I_1 - I_2) = 0$$

$$(5 + j5)I_1 - j5I_2 = 30\angle 0° \quad \text{...(i)}$$

On applying KVL to Mesh 2, we get

$$-j5(I_2 - I_1) - (2 + j3)I_2 - 6(I_2 - I_3) = 0$$

$$-j5I_1 + (8 + j8)I_2 - 6I_3 = 0 \quad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-6(I_3 - I_2) - 4I_3 - V_2 = 0$$

$$-6I_2 + 10I_3 = -V_2 \quad \text{...(iii)}$$

Writing equations in matrix form,

$$\begin{bmatrix} 5+j5 & -j5 & 0 \\ -j5 & 8+j8 & -6 \\ 0 & -6 & 10 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 30\angle 0° \\ 0 \\ -V_2 \end{bmatrix}$$

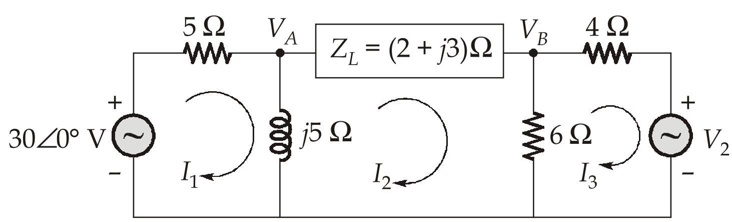
By Cramer's rule, current through the impedance Z_L is obtained as

$$I_2 = \frac{\begin{vmatrix} 5+j5 & 30\angle 0° & 0 \\ -j5 & 0 & -6 \\ 0 & -V_2 & 10 \end{vmatrix}}{\begin{vmatrix} 5+j5 & -j5 & 0 \\ -j5 & 8+j8 & -6 \\ 0 & -6 & 10 \end{vmatrix}} = 0$$

$$(5 + j5)(-6V_2) + (30)(j50) = 0$$

$$V_2 = \frac{j1500}{30+j30} = 35.36\angle 45° \text{ V}$$

Alternate:



For $I_2 = 0$, V_A must be equal to V_B.

By applying voltage division rule in both first and third loop, we get

$$V_A = \frac{30\times j5}{5+j5} \text{ Volt}; \qquad V_B = \frac{V_2\times 6}{10} \text{ Volt}$$

$\because$ $V_A = V_B$. Thus, we get

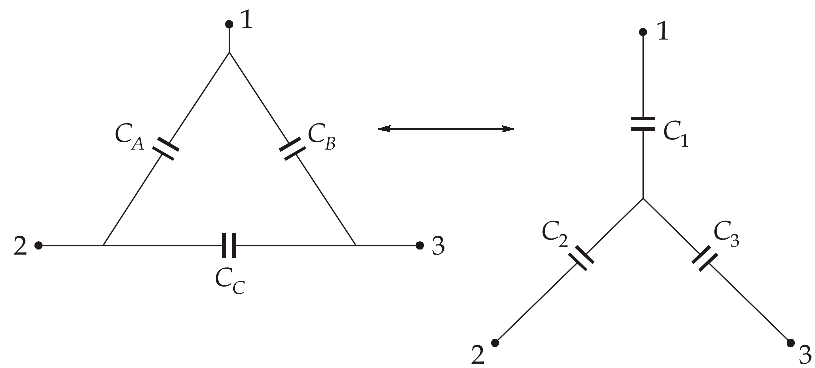
$$\frac{30\times j5}{5+5j} = \frac{V_2\times 6}{10}$$

$$V_2 = \frac{300\times j5}{6(5+j5)}$$

$$= 35.36 \angle 45° \text{ Volt}$$

Q.1 (e) Solution:

We have to convert Delta connected capacitive circuit into star connection as shown in figure below,



Now, the equivalent impedance across 1-2 in Delta and star connection are $C_{eq12} = \left[C_A + \frac{C_C \cdot C_B}{C_C + C_B}\right]$ and $C_{eq12} = \frac{C_1 \cdot C_2}{C_1 + C_2}$ respectively. As both the circuits are equivalent, hence

$$C_A + \frac{C_C \cdot C_B}{C_C + C_B} = \frac{C_1 \cdot C_2}{C_1 + C_2} \quad \text{..(i)}$$

Similarly,

$$C_{eq23} = C_C + \frac{C_A \cdot C_B}{C_A + C_B} = \frac{C_2 \cdot C_3}{C_2 + C_3} \quad \text{...(ii)}$$

and

$$C_{eq13} = C_B + \frac{C_A \cdot C_C}{C_A + C_C} = \frac{C_1 \cdot C_3}{C_1 + C_3} \quad \text{...(iii)}$$

Now on solving equation (i), we get

$$\frac{C_A(C_C + C_B) + C_C \cdot C_B}{(C_C + C_B)} = \frac{C_1 \cdot C_2}{C_1 + C_2}$$

or

$$\frac{1}{C_1} + \frac{1}{C_2} = \frac{C_C + C_B}{C_A C_C + C_A C_B + C_C C_B} \quad \text{...(iv)}$$

Similarly

$$\frac{1}{C_1} + \frac{1}{C_3} = \frac{C_A + C_C}{C_A C_C + C_A C_B + C_C C_B} \quad \text{...(v)}$$

and

$$\frac{1}{C_2} + \frac{1}{C_3} = \frac{C_A + C_B}{C_A C_C + C_A C_B + C_C C_B} \quad \text{...(vi)}$$

On subtracting (vi) from (iv), we get

$$\frac{1}{C_1} - \frac{1}{C_3} = \frac{C_C + C_B}{C_A C_C + C_A C_B + C_C C_B} - \frac{C_A + C_B}{C_A C_C + C_A C_B + C_C C_B}$$

$$\frac{1}{C_1} - \frac{1}{C_3} = \frac{C_C - C_A}{C_A C_C + C_A C_B + C_C C_B} \quad ...(vii)$$

On adding equation (v) and (vii), we get

$$\frac{2}{C_1} = \frac{2C_C}{C_A C_B + C_A C_C + C_B C_C}$$

$$C_1 = \frac{C_A C_B + C_A C_C + C_B C_C}{C_C}$$

Similarly on solving, we get

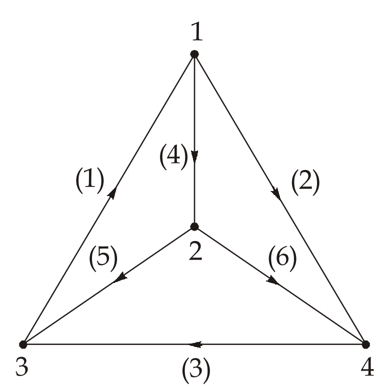
$$C_2 = \frac{C_A C_B + C_A C_C + C_B C_C}{C_B}$$

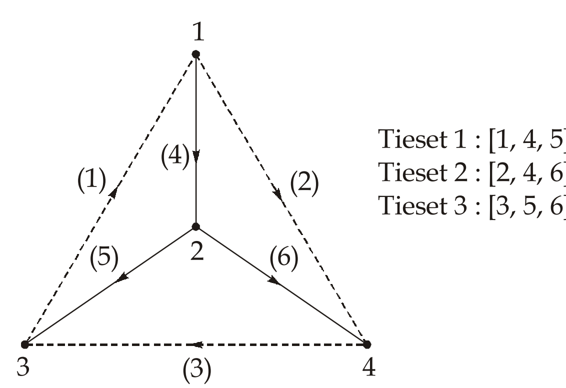
and

$$C_3 = \frac{C_A C_B + C_A C_C + C_B C_C}{C_A}$$

Q.2 (a) Solution:

The oriented graph and one of its trees are shown in figure below:





Tieset Matrix (B)

$$B = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \end{matrix}$$

The KVL equation in matrix form is given by

$$BZ_b B^T I_l = B V_s - BZ_b I_s$$

Here, $$I_s = 0,$$

Thus, $$BZ_b B^T I_l = BV_s$$

where $$Z_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad V_s = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$BZ_b = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 0 & -2 & 2 \end{bmatrix}$$

$$BZ_b B^T = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 & -2 \\ -2 & 5 & -2 \\ -2 & -2 & 5 \end{bmatrix}$$

$$BV_s = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

The KVL equation in matrix form is given by

$$\begin{bmatrix} 5 & -2 & -2 \\ -2 & 5 & -2 \\ -2 & -2 & 5 \end{bmatrix} \begin{bmatrix} I_{l_1} \\ I_{l_2} \\ I_{l_3} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Solving this matrix equation using Cramer's rule, we get

$$I_{l_1} = \frac{\begin{vmatrix} 2 & -2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 5 \end{vmatrix}}{\begin{vmatrix} 5 & -2 & -2 \\ -2 & 5 & -2 \\ -2 & -2 & 5 \end{vmatrix}} = \frac{42}{49} = \frac{6}{7}\text{A}$$

$$I_{l_2} = \frac{\begin{vmatrix} 5 & 2 & -2 \\ -2 & 0 & -2 \\ -2 & 0 & 5 \end{vmatrix}}{\begin{vmatrix} 5 & -2 & -2 \\ -2 & 5 & -2 \\ -2 & -2 & 5 \end{vmatrix}} = \frac{14}{49} = \frac{4}{7}\text{A}$$

$$I_{l_3} = \frac{\begin{vmatrix} 5 & -2 & 2 \\ -2 & 5 & 0 \\ -2 & -2 & 0 \end{vmatrix}}{\begin{vmatrix} 5 & -2 & -2 \\ -2 & 5 & -2 \\ -2 & -2 & 5 \end{vmatrix}} = \frac{14}{49} = \frac{4}{7}\text{A}$$

The branch currents are given by

$$I_b = B^T I_l$$

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 6/7 \\ 4/7 \\ 4/7 \end{bmatrix} = \begin{bmatrix} 6/7 \\ 4/7 \\ 4/7 \\ 2/7 \\ 2/7 \\ 0 \end{bmatrix}$$

Q.2 (b) Solution:

From the given description in question, we get

$$P_{CD} = P_{AB}$$

$$I_{CD}^2 R_{CD\ \text{equivalent}} = I_{AB}^2 R_{AB\ \text{equivalent}}$$

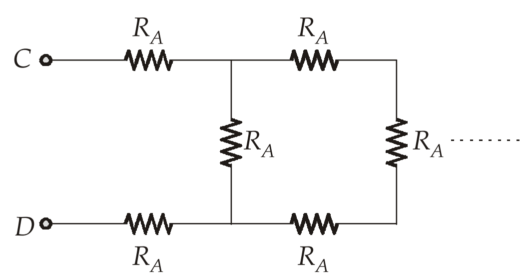
$$\because \quad I = \frac{V}{R}$$

$$\frac{V_{CD}^2}{R_{CD\text{ equivalent}}} = \frac{V_{AB}^2}{R_{AB\text{ equivalent}}}$$

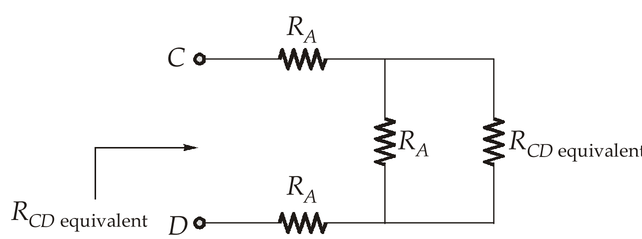
$$\because \quad V_{CD} = V_{AB} = 10\text{ V}$$

$$\therefore \quad R_{CD\text{ equivalent}} = R_{AB\text{ equivalent}}$$

Now, Equivalent resistance across CD is given as



or



$$R_{CD\text{ equivalent}} = 2R_A + (R_A \parallel R_{CD\text{ equivalent}})$$

$$R_{CD\text{ equivalent}} = 2R_A + \frac{R_A \cdot R_{CD\text{ equivalent}}}{R_A + R_{CD\text{ equivalent}}}$$

$$R_A R_{CD\text{equivalent}} + R_{CD\text{ equivalent}}^2 = 2R_A^2 + 2R_A \cdot R_{CD\text{equivalent}} + R_A \cdot R_{CD\text{equivalent}}$$

$$R_{CD\text{equivalent}}^2 - 2R_A \cdot R_{CD\text{equivalent}} - 2R_A^2 = 0$$

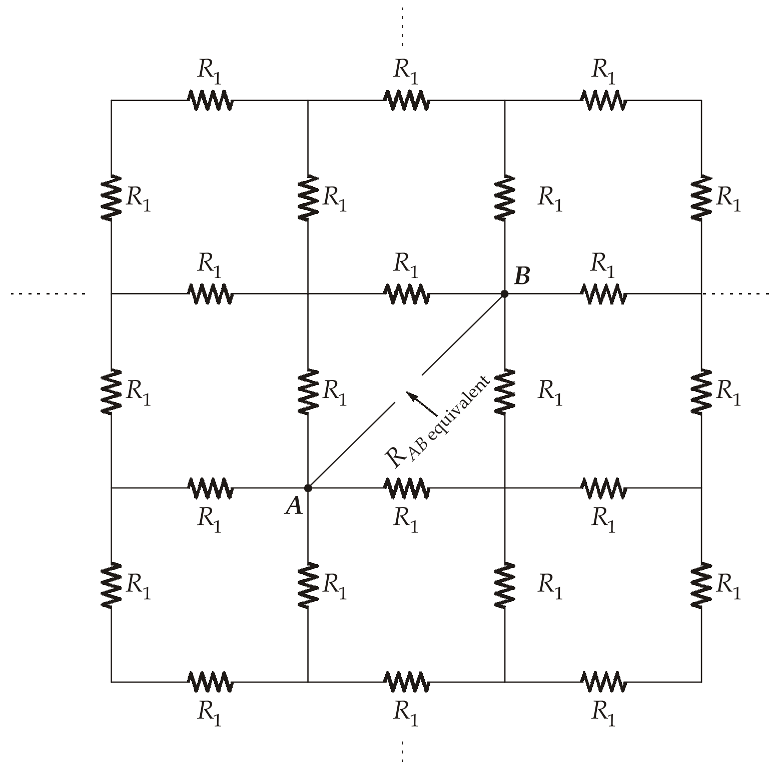
$$R_{CD\text{ equivalent}} = \frac{2R_A \pm \sqrt{4R_A^2 + 8R_A^2}}{2}$$

$$R_{CD\text{ equivalent}} = R_A \pm \sqrt{R_A^2 + 2R_A^2}$$

Since R_A cannot be negative, we get

$$R_{CD\text{ equivalent}} = (R_A + \sqrt{3}R_A)\,\Omega \qquad \text{...(i)}$$

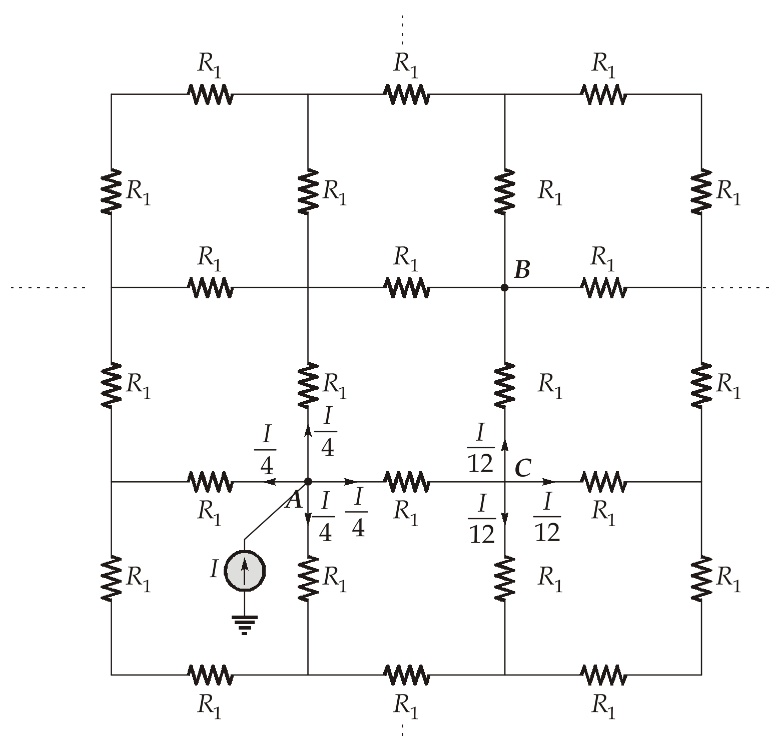
Now, equivalent resistance across A-B in network (ii) is calculated as,



Network (ii)

$R_{AB \text{ equivalent}}$:

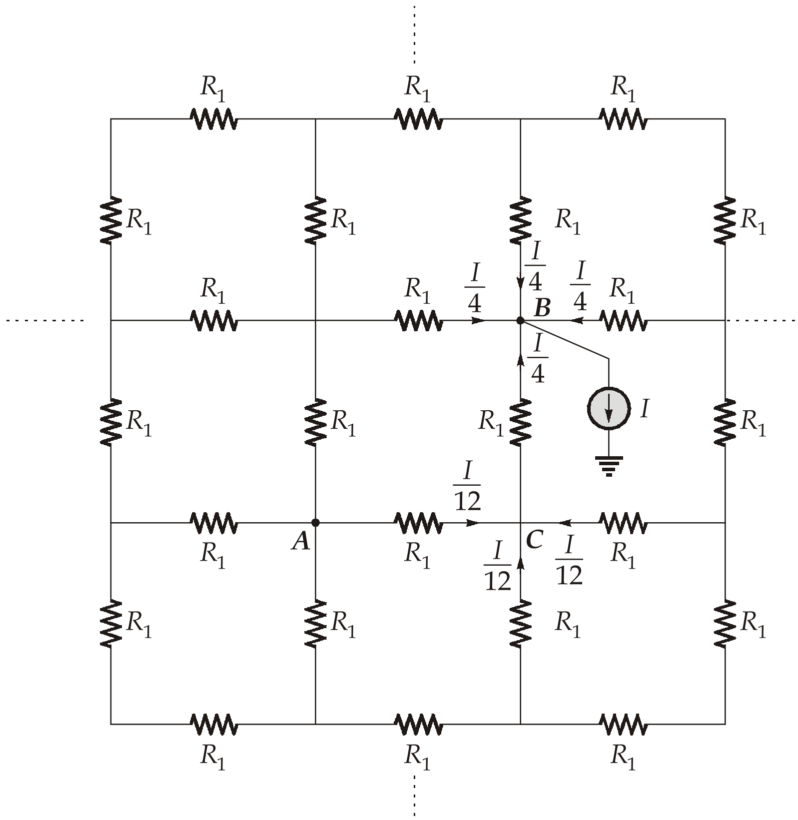
Case 1:



By KVL [$ACBA$]

$$V'_{AB} = R_1 \frac{I}{4} + R_1 \frac{I}{12} = I\left[\frac{R_1}{3}\right]$$

Case 2:



By KVL [$ACBA$]

$$V''_{AB} = R_1 \frac{I}{4} + R_1 \frac{I}{12} = I\left[\frac{R_1}{3}\right]$$

By super-position,

$$V_{AB} = V'_{AB} + V''_{AB} = \frac{2}{3} R_1 I$$

$$R_{AB\,\text{equivalent}} = \frac{2R_1}{3} \qquad \text{...(ii)}$$

$\because$

$$R_{CD\,\text{equivalent}} = R_{AB\,\text{equivalent}}$$

$$R_A + \sqrt{3}R_A = \frac{2R_1}{3}$$

$$R_1 = \left[\frac{3+3\sqrt{3}}{2}\right] R_A$$

$$R_1 = 4.10\,R_A \quad \text{or}$$
$$R_A = 0.244\,R_1$$

Now, Power delivered by source for $R_A = 10\ \Omega$ is

$$P_{\text{delivered}} = \frac{V^2}{R_{AB\,\text{equivalent}}}$$

where
$$R_{AB\,\text{equivalent}} = \frac{2R_1}{3}$$
$$R_{AB\,\text{equivalent}} = \frac{2(4.10R_A)}{3} = \frac{2\times 4.10\times 10}{3} = \frac{82}{3}\ \Omega$$
$$P_{\text{delivered}} = \frac{(10)^2\times 3}{82}$$
$$P_{\text{delivered}} = 3.66\ \text{Watt}$$

Q.2 (c) Solution:

(i) We have,

Transmission parameter matrix,

$$[T] = \begin{bmatrix} 10^{-2} & 10^2 \\ 0 & 10^{-1} \end{bmatrix}$$

Thus, we can write
$$V_1 = 0.01V_2 - 100I_2$$
$$I_1 = -0.1I_2$$

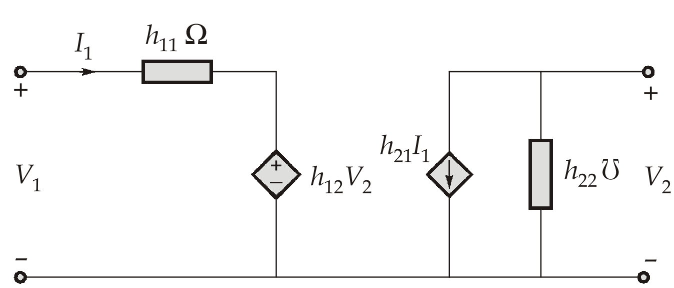
As the given equivalent circuit is similar to h-parameter equivalent circuit. Hence, h-parameter of the circuit is obtained as below:

$$I_2 = -10I_1 \quad \text{...(i)}$$

and
$$V_1 = 0.01V_2 + 1000I_1 \quad \text{...(ii)}$$

The h-parameter equations and the corresponding equivalent circuit is given by

$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$

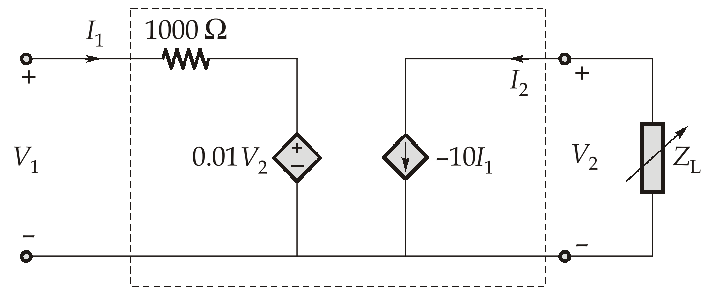


On comparing, we get

$$h_{11} = R_1 = 1000\ \Omega; \quad h_{12} = x = 0.01$$

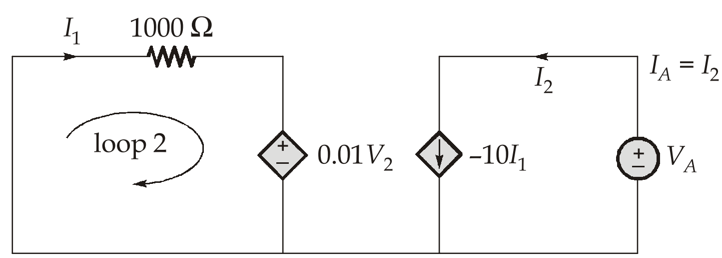
$$h_{21} = y = -10; \quad h_{22} = z = 0\ \mho$$

Now, we can redraw circuit as,



(ii) For maximum power transfer, $Z_L = Z_{th}$

***Z_{th}*: **Replacing the independent voltage source V_1 with a short-circuit and connecting voltage source V_A across the load terminals to calculate the Thevenin equivalent impedance,



$\because$ $$I_A = I_2 = -10I_1 \quad \text{...(i)}$$

On applying KVL in loop 1, we get

$$1000I_1 + 0.01V_2 = 0$$

$$1000I_1 = -0.01V_2$$

From equation (i), we get

$$I_1 = \frac{-I_A}{10}$$

We get, $$-0.01V_A = -\frac{1000}{10}I_A \quad [\because V_2 = V_A]$$

$$\frac{V_A}{I_A} = Z_{th} = \frac{+1000}{10 \times 1} \times 100$$

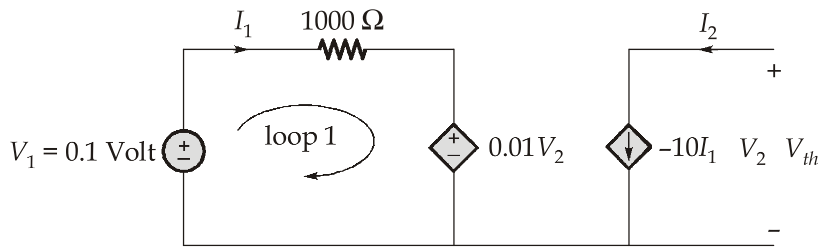
$$Z_{th} = 10\ \text{k}\Omega$$

Hence,

for maximum power transfer, $Z_L = Z_{th} = 10\ \text{k}\Omega$

(iii) Maximum power transfer to the load for $V_1 = 0.1$ V:

V_{th}:



As $I_2 = 0 = 10I_1 \quad \Rightarrow \quad I_1 = 0$

Hence, no current flow in the loop 1. It implies

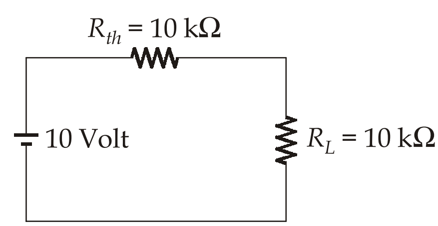
$$V_1 = 0.01\ V_2 = 0.01V_{th}$$

$$V_{th} = 100\ V_1$$

For $V_1 = 0.1$ V

$$V_{th} = 10\ V$$

Thus, Thevenin equivalent circuit can be drawn as below,



$$P_{L_{max}} = \frac{V_{th}^2}{4R_L}$$

$$P_{L_{max}} = \frac{100}{4\times10\times1000}$$

$$P_{L_{max}} = 2.5\ \text{mW}$$

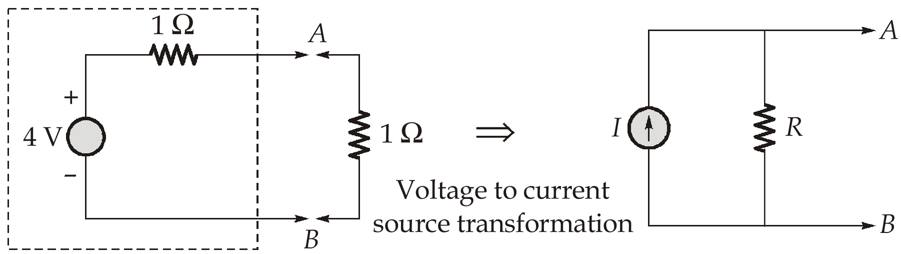
Q.3 (a) Solution:

(i) Source Transformation:

- A practical voltage source consists of an ideal voltage source in series with an internal impedance (for ideal voltage source, this impedance being zero, so that the output voltage becomes independent of the load current) while a practical current source is an ideal current source in parallel with an internal impedance (for ideal current source, this parallel impedance is infinity such that the source current does not face any branching through this internal impedance path.)

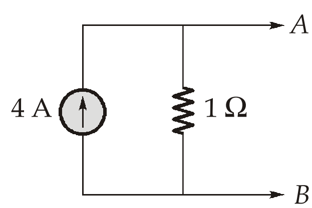
- The voltage and current sources are mutually transferable.
- For any voltage source, if the ideal voltage be V and internal resistance be R, the voltage source can be replaced by current source I with the internal resistance in parallel to the current source and the value of I is given by $I = V/R$ and vice-versa.

For the given circuit,



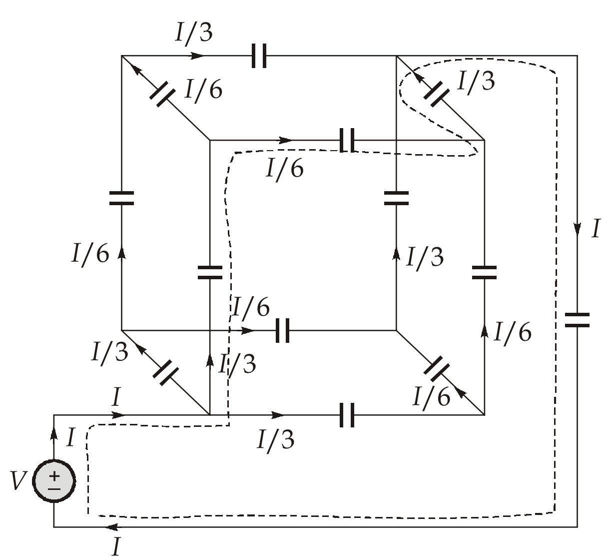
Using source transformation, we get $R = 1\ \Omega$ and $I = \frac{V}{R} = \frac{4}{1} = 4\text{ A}$

Hence,



is the valid equivalent network.

(ii) The circuit is symmetrical about A-B so that current is divided equally at nodes as shown in the figure below:



The voltage across the inductor with the current I flowing through it is given by

$$V = \frac{1}{C}\int I dt \qquad ...(i)$$

On applying KVL at the input loop,

$$V = \frac{1}{C}\int\left(\frac{I}{3}\right)dt + \frac{1}{C}\int\left(\frac{I}{6}\right)dt + \frac{1}{C}\int\left(\frac{I}{3}\right)dt + \frac{1}{C}\int I\,dt$$

$$V = \frac{11}{6C}\int I dt$$

$$V = \frac{1}{\left(\frac{6C}{11}\right)}\int I dt \qquad ...(ii)$$

On comparing (i) and (ii), we get

$$C_{equivalent} = \frac{6C}{11} \text{ Farad}$$

For C = 2 Farad,

$$C_{equivalent} = \frac{6\times 2}{11} = \frac{12}{11} \text{ Farad}$$

The energy stored by the circuit is $E = \frac{1}{2}CV^2$

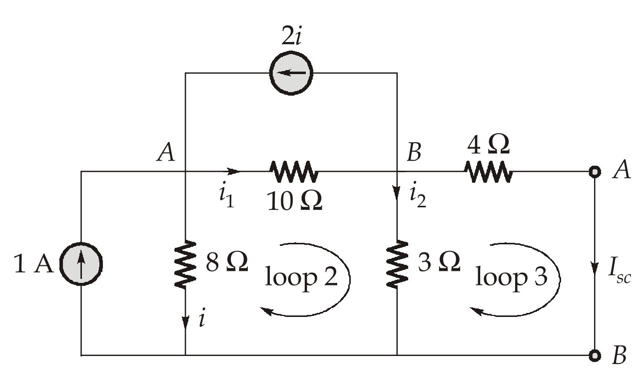
For V = 5 V, $$E = \frac{1}{2}\times\frac{12}{11}\times 5^2 = \frac{150}{11} \text{ Joules}$$

Q.3 (b) Solution:

(i) Norton equivalent circuit across *A-B*:

- **Short circuit current across *A-B*:**

On short-circuiting the terminals *A-B*, the circuit can be drawn as below,



On applying KCL at node A and node B, we get

$$i_1 + i = 2i + 1$$

$$i_1 = i + 1 \quad ...(1)$$

and

$$i_2 + I_{sc} + 2i = i_1$$

$$i_2 + I_{sc} + 2i = i + 1$$

$$i_2 = 1 - i - I_{sc} \quad ...(2)$$

On applying KVL in loop-3, we get

$$4I_{sc} - 3i_2 = 0$$

$$4I_{sc} = 3i_2$$

$$\frac{4I_{sc}}{3} = i_2 \quad ...(3)$$

Using equation (3) in (2), we get

$$\frac{4I_{sc}}{3} = 1 - i - I_{sc}$$

$$\frac{7I_{sc}}{3} = 1 - i \quad ...(4)$$

Similarly using KVL in loop 2, we get

$$10i_1 + 3i_2 - 8i = 0$$

$$10(i + 1) + 3(1 - i - I_{sc}) - 8i = 0$$

$$10i + 10 + 3 - 3i - 3I_{sc} - 8i = 0$$

$$-i + 13 = 3I_{sc}$$

$$i = 13 - 3I_{sc} \quad ...(5)$$

Using (4) and (5), we get

$$\frac{7I_{sc}}{3} = 1 - (13 - 3I_{sc})$$

$$\frac{7I_{sc}}{3} = 1 - 13 + 3I_{sc}$$

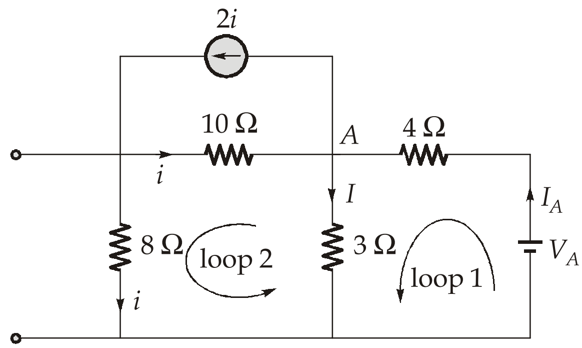
$$\frac{7I_{sc}}{3} - 3I_{sc} = -12$$

$$\frac{-2I_{sc}}{3} = -12$$

$$I_{sc} = 18 \text{ A}$$

- **Equivalent impedance (Z_{th}):**

  Assume a voltage source V_A is connected across the load terminals,



On applying KCL at node A, we get

$$I_A + i = 2i + I$$
$$I = (I_A - i)$$

On applying KVL in loop 1, we get

$$-V_A + 4I_A + 3I = 0$$
$$V_A = 4I_A + 3(I_A - i)$$
$$V_A = 7I_A - 3i \quad \text{...(i)}$$

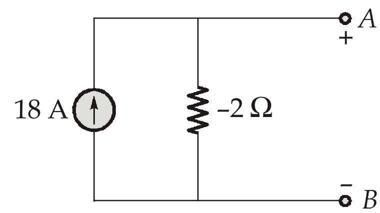
Similarly from loop 2, we get

$$-3I - 10i + 8i = 0$$
$$3I = -2i$$
$$i = \frac{3I}{-2}$$
$$i = \frac{3(I_A - i)}{-2}$$
$$-2i = 3I_A - 3i$$
$$i = 3I_A \quad \text{...(ii)}$$

Using (i) and (ii), we get

$$V_A = 7I_A - 3(3I_A)$$
$$V_A = 7I_A - 9I_A$$
$$Z_{th} = \frac{V_A}{I_A} = -2\ \Omega$$

Norton equivalent circuit across A-B is



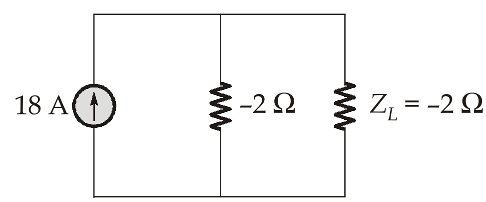
A negative Norton resistance indicates that the circuit behaves as an active source of energy rather than a passive load.

(ii) We know that,

For maximum power transfer,

$$Z_L = Z_{th}$$
$$Z_L = -2\ \Omega$$

(iii) Maximum power transfer to load,



$$P_{L\ max} = I_L^2 R_L$$
$$= (9)^2(-2)$$
$$P_{L\ max} = -162\ \text{Watt}$$

- The maximum power transferred to the load is obtained for a negative load resistance because the given network contains a dependent source, making the equivalent resistance of the source negative. The negative value of power indicates that the circuit is an active network and is capable of supplying power to load. Hence, the condition of maximum power transfer with negative load resistance is physically valid in active circuits and doesn't violate the maximum power transfer theorem.

Q.3 (c) Solution:

We know that,

When two 2-port networks N_1 and N_2 are connected in parallel, the equivalent Y-parameters of the combined network is the sum of Y-parameters of each individual 2-port network. Thus,

$$[Y] = [Y_{n1}] + [Y_{n2}]$$

and we know that $[Y] = [Z]^{-1}$. Hence,

$$[Y_{n1}] = \begin{bmatrix} 4 & 3 \\ 3 & 5 \end{bmatrix}^{-1} = \frac{1}{11}\begin{bmatrix} 5 & -3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 0.4545 & -0.272 \\ -0.272 & 0.3636 \end{bmatrix}$$

$$[Y_{n2}] = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}^{-1} = \frac{1}{8}\begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.25 \\ -0.25 & 0.375 \end{bmatrix}$$

Thus, for network N, $$[Y] = [Y_{n1}] + [Y_{n2}]$$

$$[Y] = \begin{bmatrix} 0.9545 & -0.522 \\ -0.522 & 0.7386 \end{bmatrix} \quad ...(i)$$

Now,

We know that,

Transmission parameters/$ABCD$ parameters are given as

$$V_1 = AV_2 - BI_2 \quad ...(ii)$$

$$I_1 = CV_2 - DI_2 \quad ...(iii)$$

and short-circuit parameters/Y-parameters are given as

$$I_1 = Y_{11}V_1 + Y_{12}V_2 \quad ...(iv)$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2 \quad ...(v)$$

For $I_2 = 0$,

From equation (ii) and (v), we get

$$A = \frac{V_1}{V_2} \text{ and } \frac{V_2}{V_1} = \frac{-Y_{21}}{Y_{22}}$$

Hence, $$A = \frac{V_1}{V_2} = \frac{-Y_{21}}{Y_{22}} \quad ...(1)$$

Now using equation (iii), (iv) and (v), we get

$$C = \frac{I_1}{V_2}$$

and $$I_1 = Y_{11}V_1 + Y_{12}V_2$$

$$-Y_{21}V_1 = Y_{22}V_2$$

$$V_1 = \frac{-Y_{22}}{Y_{21}}V_2$$

We can write $$I_1 = \frac{-Y_{11}Y_{22}}{Y_{21}}V_2 + Y_{12}V_2$$

$$\frac{I_1}{V_2} = C = Y_{12} - \frac{Y_{11}Y_{22}}{Y_{21}}$$

$$C = \frac{Y_{12}Y_{21} - Y_{11}Y_{22}}{Y_{21}} \quad ...(2)$$

For $V_2 = 0$,

$$B = \frac{-V_1}{I_2} \quad \text{and} \quad \frac{V_1}{I_2} = +\frac{1}{Y_{21}}$$

∴ $$B = \frac{-1}{Y_{21}} \quad ...(3)$$

Similarly, $$D = \frac{-I_1}{I_2} = \frac{-Y_{11}}{Y_{21}} \quad ...(4)$$

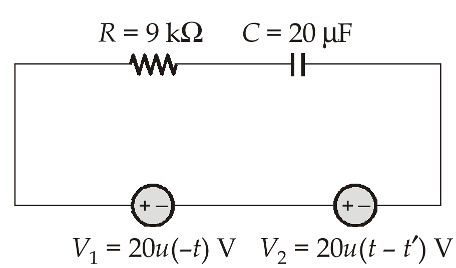
Hence, $$[T] = \begin{bmatrix} \dfrac{-Y_{21}}{Y_{22}} & -\dfrac{1}{Y_{21}} \\ \dfrac{Y_{12}Y_{21} - Y_{22}Y_{11}}{Y_{21}} & \dfrac{-Y_{11}}{Y_{21}} \end{bmatrix}$$

Now, $$[T] = \begin{bmatrix} \dfrac{0.522}{0.7386} & -\dfrac{1}{0.522} \\ \dfrac{-(0.7386\times0.9545 - 0.522\times0.522)}{-0.522} & \dfrac{-0.9545}{-0.522} \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0.707 & -1.92 \\ 0.83 & 1.83 \end{bmatrix}$$

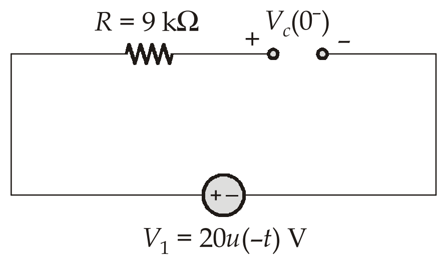
Q.4 (a) Solution:

According to given description, we can draw the circuit as



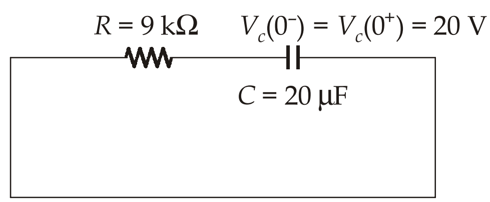
Now, for $t < 0$, since $t' > 0$, only voltage source V_1 is acting.

At $t = 0^-$, circuit is in the steady state. The capacitor is fully charged and acts as open circuit.



$$V_c(0^-) = 20 \text{ Volt}$$

Now, for $0 < t < t'$; When all the supplies are disabled, the voltage across the capacitor discharges through the resistance R. Since the capacitor does not allow sudden change in voltage, $V_c(0^-) = V_c(0^+) = 20$ V.



The voltage across the capacitor is given by

$$V_c(t) = V_c(\infty) + (V_c(0^+) - V_c(\infty))e^{-t/\tau}$$

where $V_c(\infty) = 0 \text{ V};\ V_c(0^+) = 20 \text{ V}$

and $\tau = RC = 9 \times 10^{+3} \times 20 \times 10^{-6}$

$$\tau = 180 \text{ msec}$$

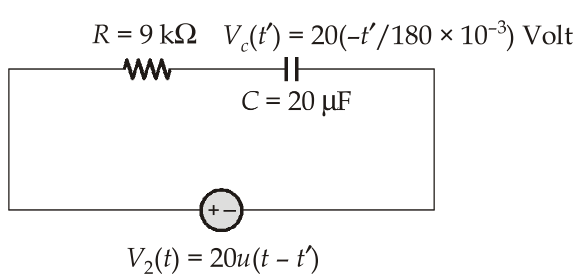
$$V_c(t) = 20e^{(-t/180 \times 10^{-3})} \text{ Volt}$$

At $t = t'$,

$$V_c(t') = 20e^{(-t'/180 \times 10^{-3})} \text{Volt}$$

$$V_c(t') = 20e^{-(5.55t')} \text{Volt}$$

Now, at $t = t'$, the voltage source $V_2(t)$ is acting. Thus, the circuit can be drawn as below:



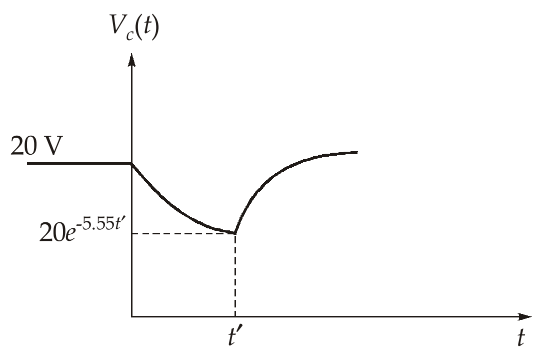
For $t > t'$, the capacitor charges through resistance R. The voltage across the capacitor is given by

$$V_c(t) = V_c(\infty) + (V_c(t') - V_c(\infty)e^{-t/\tau}$$

$$V_c(t) = 20 + (20e^{-t'/180\times10^{-3}} - 20)e^{-(t-t')/\tau}$$

where $\tau = 180$ msec

Thus,



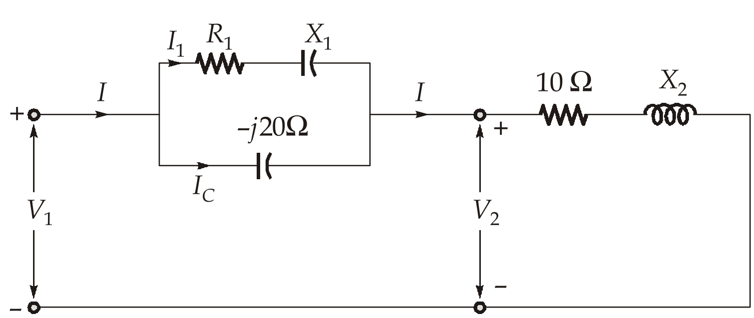
$$V_c(t) = 20 + (20e^{-t'/180\times10^{-3}} - 20)e^{\frac{-(t-t')}{180\times10^{-3}}}\text{Volt}$$

We get,

$$V_c(t) = \begin{cases} 20\text{ V} & ; \quad t < 0 \\ 20e^{-(5.55t)}\text{Volt} & ; \quad 0 < t < t' \\ 20 + (20e^{-5.55t'} - 20)e^{-(t-t')5.55}\text{Volt}\ ; & \quad t > t' \end{cases}$$

Q.4 (b) Solution:

The given circuit is,

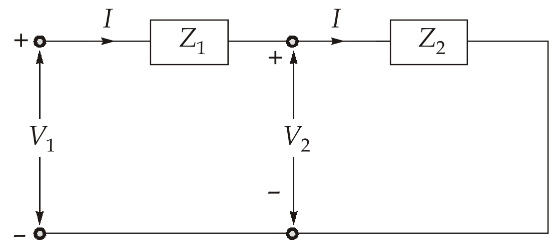


Given: $|V_1| = 200$ V and $V_2 = 200\angle0°$ V

also, $|I| = 12$ A

$P_T = 1.8$ kW

Now, the circuit can be redrawn as



where, $$\frac{1}{Z_1} = \frac{1}{Z} + \frac{1}{-j20\Omega} \quad \text{with } Z = R_1 - jX_1$$

and, $$Z_2 = R_2 + jX_2$$
$$= 10 + jX_2 = Z_2 \angle\theta_2$$

$\therefore$ $$I = \frac{V_2}{Z_2} = \frac{200\angle 0°}{Z_2\angle\theta_2}$$

Thus, $$|I| = \frac{200}{Z_2} = 12$$

Hence, $$Z_2 = \frac{200}{12} = 16.67\Omega$$

$\because$ $$|Z_2| = \sqrt{10^2 + X_2^2}$$

$\therefore$ $$(16.67)^2 = 100 + X_2^2$$

or $$X_2^2 = 177.77$$

or $$X_2 = \sqrt{177.77} = 13.33\Omega$$

and $$\theta_2 = \tan^{-1}\left(\frac{X_2}{10}\right) = \tan^{-1}\left(\frac{13.33}{10}\right) = 53.13°$$
$$\theta_2 = 53.13°$$

Thus, angle between I and V_2 is 53.13°

$\therefore$ $$I = 12\angle -53.13° \text{A}$$

Now, $$P_T = V_1 \times I \cos\theta_T$$

where, $$\cos\theta_T = \frac{P_T}{V_1 \times I} = \frac{1800}{200 \times 12} \quad \text{(given)}$$
$$\cos\theta_T = 0.75$$

or $$\theta_T = 41.4°$$

$\because$ This is the angle between current I and voltage V_1

$\therefore$ $$V_1 = 200\angle(-53.13 + 41.4)°$$

or $$V_1 = 200\angle -11.73° \text{ V}$$

and $$V_{z1} = V_1 - V_2 = 200\angle -11.73° - 200\angle 0°$$
$$= 195.82 - 40.659j - 200 = -4.18 - 40.659j = 40.87\angle -95.86°$$

Current through capacitor ($-j20\ \Omega$) is

$$I_C = \frac{V_{z1}}{jX_c} = \frac{40.87\angle -95.86°}{20\angle -90°} = 2.0436\ \angle -5.86°\ \text{A}$$

Now, current through R_1 and X_1 are

$$I_1 = I - I_C = 12\angle -53.13° - 2.0436\ \angle -5.86°$$

$$I_1 = 10.718\angle -61.18°\ \text{A}$$

$$\therefore \quad Z_1 = \frac{V_{z1}}{I_1} = \frac{40.87\angle -95.86°}{10.718\angle -61.18°}$$

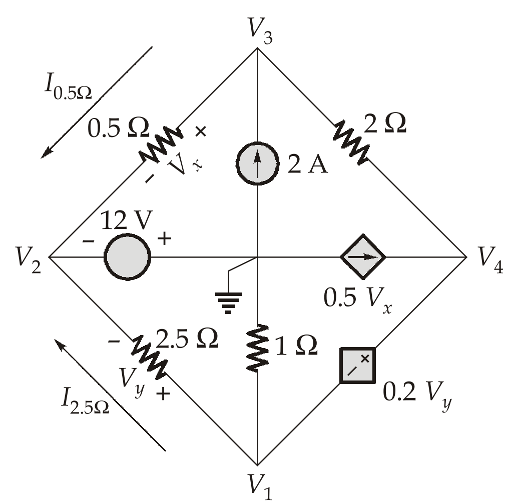
$$= 3.813\ \angle -34.68° = 3.135 - 2.169j = R_1 - jX_1$$

$$\therefore \quad R_1 = 3.135\ \Omega$$

$$X_1 = 2.169\ \Omega$$

Q.4 (c) Solution:

We have,



On applying KVL at each node, we get

$$V_3 - V_2 = V_x \quad \text{...(i)}$$

$$V_2 = -12\ \text{V} \quad \text{...(ii)}$$

Thus, $$V_3 = V_x - 12$$

$$V_1 - V_2 = V_y \quad \text{...(iii)}$$

On applying KCL at node V_3, we get

$$\frac{V_3 - V_2}{0.5} + \frac{V_3 - V_4}{2} = 2$$

$$\because \quad V_2 = -12\ \text{V}$$

$$\frac{V_3+12}{0.5}+\frac{V_3-V_4}{2} = 2$$

$$4V_3 + 48 + V_3 - V_4 = 4$$

$$5V_3 - V_4 = -44 \quad \text{...(iv)}$$

On applying KCL at supernode 1-4, we get

$$\frac{V_1-V_2}{2.5}+\frac{V_1}{1}+\frac{V_4-V_3}{2} = 0.5V_x$$

$$\frac{V_1-V_2}{2.5}+V_1+\frac{V_4-V_3}{2} = 0.5(V_3 + 12)$$

$$\frac{V_1+12}{2.5}+V_1+\frac{V_4-V_3}{2} = 0.5V_3 + 6$$

$$2V_1 + 24 +5V_1 + 2.5V_4 - 2.5V_3 = 2.5V_3 + 30$$

$$7V_1 - 5V_3 + 2.5\ V_4 = 6 \quad \text{...(v)}$$

Adding (iv) and (v) we get,

$$7V_1 + 1.5V_4 = -38 \quad \text{...(vi)}$$

$$V_4 - V_1 = 0.2V_y$$

$$V_4 - V_1 = 0.2(V_1 - V_2) \quad \text{...using equation (iii)}$$

$$V_4 - V_1 = 0.2V_1 - 0.2\ V_2$$

$$1.2V_1 - 0.2V_2 - V_4 = 0$$

$$1.2V_1 + 2.4 - V_4 = 0$$

$$1.2V_1 - V_4 = -2.4 \quad \text{...(vii)}$$

On solving (vi) and (vii), we get

$$V_1 = -4.72\ \text{V}$$

$$V_4 = -3.27\ \text{Volt}$$

$$V_2 = -12\ \text{Volt}$$

$$V_3 = -9.443\ \text{Volt} \quad \text{...using equation (iv)}$$

Thus,

Current through 0.5 Ω resistor, $I_{0.5\,\Omega} = \frac{V_3-V_2}{0.5} = \frac{-9.443+12}{0.5} = 5.114$ A

Current through 2.5 Ω resistor, $I_{2.5\,\Omega} = \frac{V_1-V_2}{2.5} = \frac{-4.72+12}{2.5} = 2.912$ A

Section B : Signals and Systems

Q.5 (a) Solution:

(i) Commutative property of convolution

$$x(n) * h(n) = h(n) * x(n)$$

We know that,

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

Let $l = n - k$

$\therefore$ $k = n - l$

When $k = -\infty$, $l = \infty$

When $k = \infty$, $l = -\infty$

$$\therefore \quad x(n) * h(n) = \sum_{l=\infty}^{-\infty} x(n-l)h(l) = \sum_{l=-\infty}^{\infty} h(l)x(n-l)$$

$$\therefore \quad x(n) * h(n) = h(n) * x(n) \quad \text{...(i)}$$

Distributive property of convolution,

$$h(n) * [x_1(n) + x_2(n)] = h(n) * x_1(n) + h(n) * x_2(n)$$

From (i), we know that,

$$x(n) * h(n) = h(n) * x(n)$$

$$\therefore \quad x(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

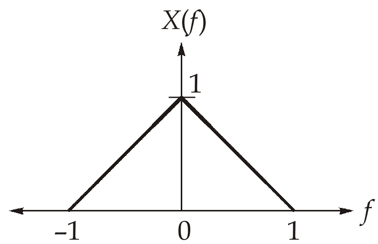
$$h(n) * [x_1(n) + x_2(n)] = \sum_{k=-\infty}^{\infty} h(k)[x_1(n-k)+x_2(n-k)]$$

$$= \sum_{k=-\infty}^{\infty} h(k)x_1(n-k) + \sum_{k=-\infty}^{\infty} h(k)x_2(n-k)$$

$$\therefore \quad h(n) * [x_1(n) + x_2(n)] = h(n) * x_1(n) + h(n) * x_2(n)$$

(ii) 1. Given;

$$X(f) = \begin{cases} 1-|f| & ; \quad |f| \le 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$



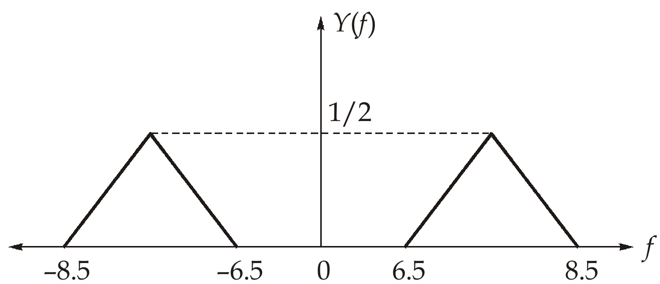
$$y(t) = x(t) \cdot m(t) = x(t) \cos(15\pi t)$$

$$= \frac{1}{2}x(t)e^{j15\pi t} + \frac{1}{2}x(t)e^{-j15\pi t}$$

$\therefore$ $$y(t) = \frac{1}{2}x(t)e^{j2\pi\frac{15}{2}t} + \frac{1}{2}x(t)e^{-j2\pi\frac{15}{2}t}$$

Using the frequency shifting property of Fourier Transform, we get

$$Y(f) = \frac{1}{2}X(f+7.5) + \frac{1}{2}X(f-7.5)$$

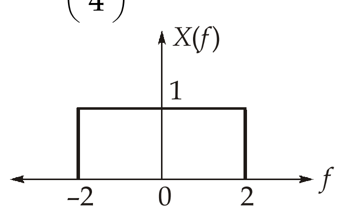


2. $$y(t) = x(t) \cdot m(t) = x(t) \cdot \cos 2\pi t = \frac{1}{2}x(t)e^{j2\pi t} + \frac{1}{2}x(t)e^{-j2\pi t}$$

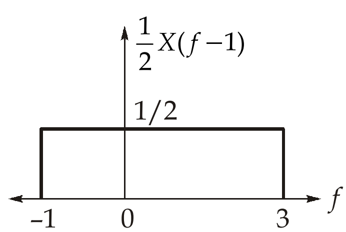
$\therefore$ $$y(f) = \frac{1}{2}X(f-1) + \frac{1}{2}X(f+1)$$

Now, $$X(f) = \text{rect}\left(\frac{f}{4}\right)$$

$\therefore$ $X(f) \rightarrow$



$X(f) \rightarrow$



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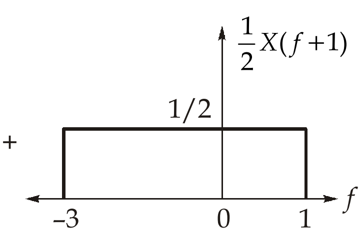
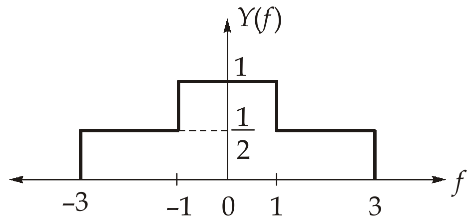




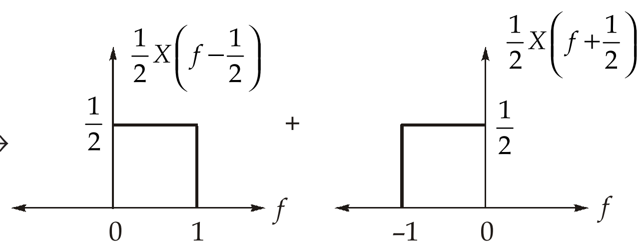
3.

$$Y(t) = x(t)\cdot m(t) = x(t)\cdot \cos \pi t = \frac{1}{2}x(t)e^{j\pi t} + \frac{1}{2}x(t)e^{-j\pi t}$$

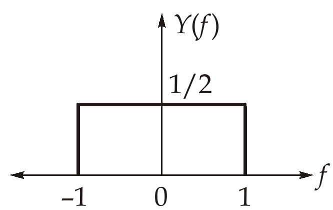
$$y(t) = \frac{1}{2}x(t)e^{j2\pi\frac{1}{2}t} + \frac{1}{2}x(t)e^{-j2\pi\frac{1}{2}t}$$

$$\therefore \quad Y(f) = \frac{1}{2}\left[X\left(f-\frac{1}{2}\right)+X\left(f+\frac{1}{2}\right)\right]$$

$Y(f) \rightarrow$



$\therefore \quad Y(f) \rightarrow$



Q.5 (b) Solution:

The complex exponential Fourier series representation of a signal $f(t)$ is given by

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}, \text{ where } \omega_0 = \frac{2\pi}{T}$$

The given signal $f(t)$ over the interval $(0, T)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{3}{4+(n\pi)^2} e^{jn\pi t}$$

(i) Comparing the above two equations, we get

$$c_n = \frac{3}{4+(n\pi)^2}$$

and

$$e^{jn\frac{2\pi}{T}t} = e^{jn\pi t}$$

Hence, $$\frac{2\pi}{T} = \pi, \quad \text{i.e. } T = 2$$

(ii) When $n = 3$, the component of $f(t)$ will be

$$c_3 = \frac{3}{4+(3\pi)^2}e^{j3\pi t} = \frac{3}{4+(3\pi)^2}[\cos 3\pi t + j\sin 3\pi t]$$

Similarly, when $n = -3$, the component of $f(t)$ will be

$$c_{-3} = \frac{3}{4+(-3\pi)^2}e^{-j3\pi t} = \frac{3}{4+(3\pi)^2}[\cos 3\pi t - j\sin 3\pi t]$$

Therefore, $$c_3 + c_{-3} = \frac{6}{4+(3\pi)^2}\cos 3\pi t$$

Hence, when one of the components of $f(t)$ is $A\cos 3\pi t$, the value of A is

$$A = \frac{6}{4+(3\pi)^2}$$

(iii) According to Parseval's theorem, the total power of the signal,

$$P_t = \sum_{n=-\infty}^{\infty}|c_n|^2 = \sum_{n=-\infty}^{\infty}\left|\frac{3}{4+(n\pi)^2}\right| \approx 0.669 \quad \text{(Given)}$$

The power contained by the signal $f(t)$ upto the first four harmonics is given by

$$P = |c_0|^2 + 2\left[|c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2\right]$$

$$= \left|\frac{3}{4}\right|^2 + 2\left[\left|\frac{3}{4+(\pi)^2}\right|^2 + \left|\frac{3}{4+(2\pi)^2}\right|^2 + \left|\frac{3}{4+(3\pi)^2}\right|^2 + \left|\frac{3}{4+(4\pi)^2}\right|^2\right]$$

$$= 0.5625 + 0.0935 + 9.52 \times 10^{-3} + 2.088 \times 10^{-3} + 6.866 \times 10^{-4}$$

$$= 0.66836$$

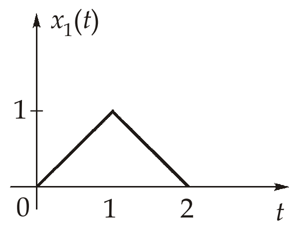
Therefore, power as a percentage of total power, the power contained by the signal $f(t)$ upto the first four harmonics is given by

$$\frac{P}{P_t} \times 100 = \frac{0.66836}{0.669} \times 100 = 99.9\%$$

Hence, 99.9% of the total power is contained by the signal $f(t)$ upto the first four harmonics.

Q.5 (c) Solution:

(i) From the figure given in the question, the first cycle of $x(t)$ is as shown in the figure below:



We know that the Laplace transform of a causal periodic signal is given by

$$x(t) = x_1(t) + x_1(t - T_0) + x_1(t - 2T_0) + ...$$

Using time shifting property of Laplace Transform,

$$X(s) = X_1(s)\left[1 + e^{-sT_0} + e^{-2sT_0} + ...\right]$$

$$X(s) = \frac{X_1(s)}{1 - e^{-sT_0}}$$

where $X_1(s)$ is the Laplace transform of the first cycle $x_1(t)$ of the causal periodic signal $x(t)$. The given sawtooth wave is a causal periodic signal with period $T_0 = 2$. The first cycle $x_1(t)$ of $x(t)$ is as shown in Figure (b). Using the ramp function $r(t)$, $x_1(t)$ can be written as

$$x_1(t) = r(t) - 2r(t - 1) + r(t - 2)$$

Taking the Laplace transform of the above equation, we obtain

$$L[x_1(t)] = L\left[r(t)\right] - 2L\left[r(t-1)\right] + L\left[r(t-2)\right]$$

$$X_1(s) = \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s} = \frac{1}{s^2}\left[1 - 2e^{-s} + e^{-2s}\right] = \frac{1}{s^2}(1 - e^{-s})^2$$

Substituting $X_1(s)$ and $T_0 = 2$ in the expression of $X(s)$, we get

$$X(s) = \frac{X_1(s)}{1 - e^{-sT_0}} = \frac{1}{s^2}\frac{(1 - e^{-s})^2}{1 - e^{-2s}} = \frac{1}{s^2}\frac{(1 - e^{-s})^2}{(1 + e^{-s})(1 - e^{-s})}$$

$$= \frac{1}{s^2}\frac{1 - e^{-s}}{1 + e^{-s}} = \frac{1}{s^2}\frac{e^{-s/2}(e^{s/2} - e^{-s/2})}{e^{-s/2}(e^{s/2} + e^{-s/2})}$$

$$X(s) = \frac{1}{s^2}\left(\frac{e^{s/2} - e^{-s/2}}{e^{s/2} + e^{-s/2}}\right) = \frac{1}{s^2}\tanh\left(\frac{s}{2}\right)$$

(ii) Given,

$$x(n) = \text{Re}\left[e^{jn\pi/12}\right] + \text{Im}\left[e^{jn\pi/18}\right]$$

$$= \text{Re}\left[\cos\left(\frac{n\pi}{12}\right) + j\sin\left(\frac{n\pi}{12}\right)\right] + \text{Im}\left[\cos\left(\frac{n\pi}{18}\right) + j\sin\left(\frac{n\pi}{18}\right)\right]$$

$$x(n) = \cos\left(\frac{n\pi}{12}\right) + \sin\left(\frac{n\pi}{18}\right)$$

Period of $\cos\left(\frac{n\pi}{12}\right), T_1 = \frac{2\pi}{\pi/12} = 2\times 12 = 24$

Period of $\sin\left(\frac{n\pi}{18}\right), T_2 = \frac{2\pi}{\pi/18} = 2\times 18 = 36$

For a discrete-time signal, if the individual signals are periodic, their sum will be surely periodic.

So overall period of $x(n)$ will be LCM of individual periods, T_1 and T_2.

$\therefore \quad N = \text{LCM}(24, 36)$

$\therefore \quad N = 72$

Q.5 (d) Solution:

To perform circular convolution of $x_1(n)$ and $x_2(n)$ using DFT and IDFT, we find $X_1(k)$ (DFT of $x_1(n)$), $X_2(k)$ (DFT of $x_2(n)$), then computing $X_3(k) = X_1(k)\, X_2(k)$ and then taking IDFT of $X_3(k)$ to get $x_3(n)$.

The 4-point DFT of $x_1(n)$ is:

$$\text{DFT}\{x_1(n)\} = X_1(k) = \sum_{n=0}^{3} x_1(n)\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, 2, 3$$

$$= x_1(0)\, e^0 + x_1(1)\, e^{-j\frac{2\pi}{4}k} + x_1(2)\, e^{-j\frac{4\pi}{4}k} + x_1(3)\, e^{-j\frac{6\pi}{4}k}$$

$$= 1 + 2e^{-j\frac{\pi}{2}k} + e^{-j\pi k} + 2e^{-j\frac{3\pi}{2}k}$$

When $k = 0$, $\quad X_1(0) = 1 + 2 + 1 + 2 = 6$

When $k = 1$, $\quad X_1(1) = 1 + 2e^{-j\frac{\pi}{2}} + e^{-j\pi} + 2e^{-j\frac{3\pi}{2}} = 1 - j2 - 1 + j2 = 0$

When $k = 2$, $\quad X_1(2) = 1 + 2e^{-j\pi} + e^{-j2\pi} + 2e^{-j3\pi} = 1 - 2 + 1 - 2 = -2$

When $k = 3$, $X_1(3) = 1+2e^{-j\frac{3\pi}{2}} + e^{-j3\pi} + 2e^{-j\frac{9\pi}{2}} = 1 + j2 - 1 - j2 = 0$

$\therefore$ $X_1(k) = \{6, 0, -2, 0\}$

The 4-point DFT of $x_2(n)$ is:

$$\text{DFT}\{x_2(n)\} = X_2(k) = \sum_{n=0}^{3} x_2(n)\, e^{-j\frac{2\pi}{4}nk}, \; k = 0, 1, 2, 3$$

$$= x_2(0)\, e^0 + x_2(1)\, e^{-j\frac{\pi}{2}k} + x_2(2)\, e^{-j\pi k} + x_2(3)\, e^{-j\frac{3\pi}{2}k}$$

$$= 4 + 3e^{-j\frac{\pi k}{2}} + 2e^{-j\pi k} + 1e^{-j\frac{3\pi}{2}k}$$

When $k = 0$, $X_2(0) = 4 + 3 + 2 + 1 = 10$

When $k = 1$, $X_1(1) = 4 + 3e^{-j\frac{\pi}{2}} + 2e^{-j\pi} + e^{-j\frac{3\pi}{2}} = 4 - j3 - 2 + j = 2 - j2$

When $k = 2$, $X_1(2) = 4 + 3e^{-j\pi} + 2e^{-j2\pi} + e^{-j3\pi} = 4 - 3 + 2 - 1 = 2$

When $k = 3$, $X_1(3) = 4 + 3e^{-j\frac{3\pi}{2}} + 2e^{-j3\pi} + e^{-j\frac{9\pi}{2}} = 4 + j3 - 2 - 1 - j = 2 + j2$

$\therefore$ $X_2(k) = \{10, 2 - j2, 2, 2 + j2\}$

Let, $X_2(k)$ be the product of $X_1(k)$ and $X_2(k)$.

$\therefore$ $X_3(k) = X_1(k)\, X_2(k) = \{6, 0, -2, 0\}\, \{10, 2 - j2, 2, 2 + j2\}$

$= \{60, 0, -4, 0\}$

$\therefore$ $$x_3(n) = \text{IDFT}\{X_3(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k) e^{j\frac{2\pi}{N}nk}; \quad n = 0, 1, 2, 3$$

$$= \frac{1}{4}\left\{X_3(0)\, e^0 + X_3(1)\, e^{j\frac{2\pi}{4}n} + X_3(2)\, e^{j\frac{4\pi}{4}n} + X_3(3)\, e^{j\frac{6\pi}{4}n}\right\}$$

$$= \frac{1}{4}\{60 - 4e^{j\pi n}\}$$

When $n = 0$,

$$x_3(0) = \frac{1}{4}(60 - 4) = 14$$

When $n = 1$,

$$x_3(1) = \frac{1}{4}(60 - 4e^{j\pi}) = \frac{1}{4}(60 + 4) = 16$$

When $n = 2$,

$$x_3(2) = \frac{1}{4}(60 - 4e^{j2\pi}) = \frac{1}{4}(60 - 4) = 14$$

When $n = 3$,

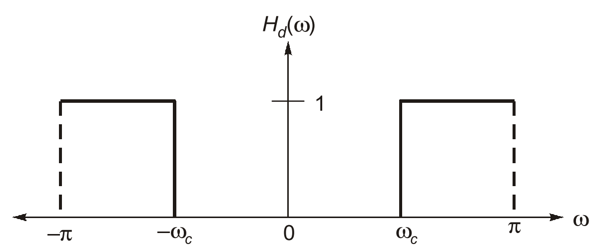
$$x_3(3) = \frac{1}{4}(60 - 4e^{j3\pi}) = \frac{1}{4}(60 + 4) = 16$$

$\therefore$ $$x_3(n) = \{14, 16, 14, 16\}$$

Q.5 (e) Solution:

Given, $f_c = 250$ Hz therefore, $\omega_c = \frac{2\pi f_c}{f_s} = \frac{\pi}{2}$ and

Normalised cut-off frequency $(f_c) = \frac{250}{1000} = \frac{1}{4}$.



Frequency response of a high pass filter

Now,
$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{-\pi} H_d(\omega)e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} 1\times e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} 1\times e^{j\omega n}d\omega$$

$$h_d(n) = \frac{1}{2\pi}\left[\left(\frac{e^{j\omega n}}{jn}\right)_{-\pi}^{-\omega_c} + \left(\frac{e^{j\omega n}}{jn}\right)_{\omega_c}^{\pi}\right] = \frac{1}{n\pi}\left[\frac{e^{-j\omega_c n}}{2j} - \frac{e^{j\omega_c n}}{2j}\right]$$

$$h_d(n) = \frac{1}{n\pi}\left[\frac{e^{-j\omega_c n}}{2j} - \frac{e^{-j\omega_c n}}{2j}\right] = -\frac{\sin\omega_c n}{n\pi}$$

For $n = 0$,
$$h_d(0) = \frac{1}{2\pi}[(-\omega_c + \pi) + (\pi - \omega_c)]$$

$$= 1 - 2f_c = \frac{1}{2} \text{ (by evaluating above integral for } n = 0)$$

Thus, we have

$$h_d(n) = \begin{cases} 1/2 & ; \quad n=0 \\ -\dfrac{\sin(n\pi/2)}{n\pi} & ; \quad n \neq 0 \end{cases}$$

For a given filter length of $M = 7$, we have $\dfrac{M-1}{2} = 3$.

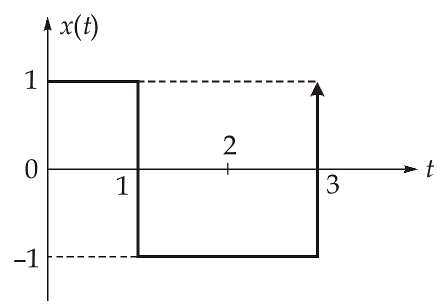
Thus, coefficients of a suitable window $\omega(n)$ and $h_d(n)$ can be computed for $-3 \le n \le 3$.

Note: The resultant $h(n) = h_d(n)\,\omega(n)$ will not be causal. However, $h(n)$ can be made causal by right shift of 3 units. (Such that $h(n) = 0$ for $n < 0$).

Q.6 (a) Solution:

(i) Part 1:

Given waveform of $x(t)$ is



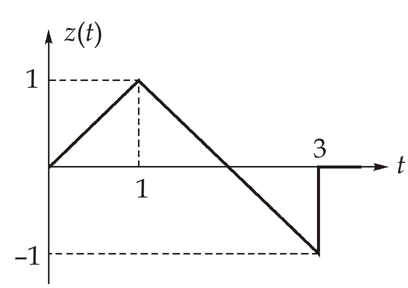
We can express $x(t)$ as:

$$x(t) = u(t) - 2u(t-1) + u(t-3) + \delta(t-3)$$

Let $$z(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

then $$z(t) = \int_{-\infty}^{t} u(\tau)d\tau - 2\int_{-\infty}^{t} u(\tau-1)d\tau + \int_{-\infty}^{t} u(\tau-3)d\tau + \int_{-\infty}^{t} \delta(\tau-3)d\tau$$

$$z(t) = r(t) - 2r(t-1) + r(t-3) + u(t-3)$$



Part 2:

We can express $y(t)$ as,

$$y(t) = u(t) - \sum_{n=1}^{\infty} \delta(t-n)$$

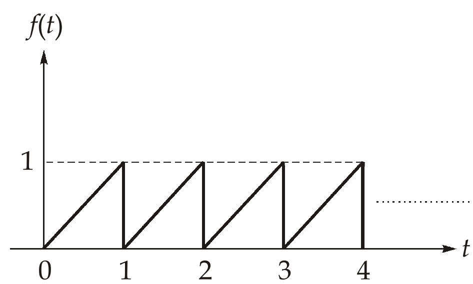
Let

$$f(t) = \int_{-\infty}^{t} y(\tau)d\tau$$

$$f(t) = \int_{-\infty}^{t} u(\tau)d\tau - \sum_{n=1}^{\infty}\left[\int_{-\infty}^{t} \delta(t-n)d\tau\right]$$

$$f(t) = r(t) - \sum_{n=1}^{\infty} u(t-n)$$

$$= r(t) - u(t-1) - u(t-2) - u(t-3) - ....$$



(ii) Given :

$$K_1 = \frac{1}{2}, K_2 = \frac{1}{4}$$

The transfer function of a lattice filter of order 'm' is given by

$$A_m(z) = A_{m-1}(z) + K_m z^{-m} A_{m-1}(z^{-1})$$

with initial condition, $A_0(z) = 1$

We have,

$$A_1(z) = A_0(z) + K_1 z^{-1} A_0(z^{-1}) = 1 + \frac{1}{2}z^{-1} \times 1$$

$$= 1 + \frac{1}{2}z^{-1} \qquad (\because A_0(z) = 1; \text{ thus } A_0(z^{-1}) = 1)$$

$$A_2(z) = A_1(z) + K_2 z^{-2} A_1(z^{-1})$$

$$= \left(1 + \frac{1}{2}z^{-1}\right) + \frac{1}{4}z^{-2}\left(1 + \frac{1}{2}z\right)$$

$$\left(\because A_1(z^{-1}) = A_1(z)\Big|_{z \to z^{-1}} = 1 + \frac{1}{2}z\right)$$

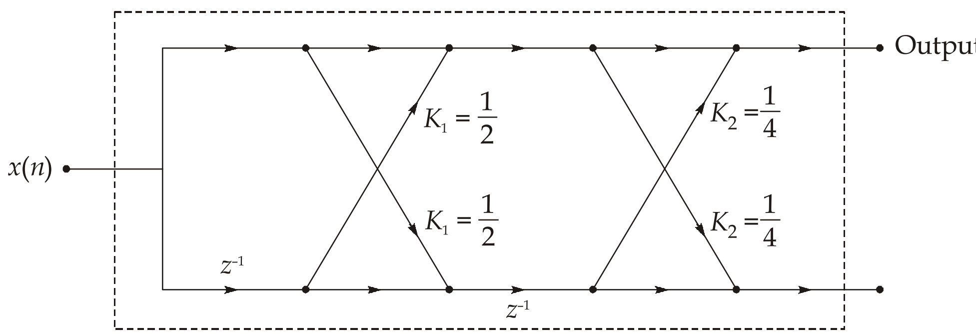
$$= 1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}+\frac{1}{8}z^{-1}$$

$$A_2(z) = 1+\frac{5}{8}z^{-1}+\frac{1}{4}z^{-2}$$

We have, $$A_2(z) = \sum_{i=0}^{2} a_2(i)z^{-1}$$

thus impulse response,

$$a_2(0) = 1,\ a_2(1) = \frac{5}{8},\ a_2(2) = \frac{1}{4}$$



Q.6 (b) Solution:

(i) 1.

$$X(s) = \frac{1}{s^2}\frac{d}{ds}\left[\frac{e^{-3s}}{s}\right], \text{ROC}: \sigma > 0$$

$$= \frac{1}{s^2}\left[\frac{-3e^{-3s}s - 1e^{-3s}}{s^2}\right]$$

$$= \frac{-3}{s^3}e^{-3s} - \frac{e^{-3s}}{s^4};\ \sigma > 0 \quad \text{...(i)}$$

Using the Multiplication by power of 't' property,

$$t^n u(t) \rightleftharpoons \frac{n!}{s^{n+1}}, \sigma > 0$$

Put $n = 2$: $$t^2 u(t) \rightleftharpoons \frac{2!}{s^3} = \frac{2}{s^3}, \sigma > 0$$

$$\frac{t^2}{2}\cdot u(t) \rightleftharpoons \frac{1}{s^3}, \sigma > 0$$

Using the time-shifting property of Laplace Transform,

$$\frac{-3(t-3)^2}{2}u(t-3) \rightleftharpoons \frac{-3e^{-3s}}{s^3}, \sigma > 0$$

Put $n = 3$: $t^3 u(t) \rightleftharpoons \frac{3!}{s^4},\ \sigma > 0$

$$\frac{t^3}{6}u(t) \rightleftharpoons \frac{1}{s^4}, \sigma > 0$$

Using the time-shifting property of Laplace Transform,

$$\frac{(t-3)^3}{6}u(t-3) \rightleftharpoons \frac{e^{-3s}}{s^4}, \sigma > 0$$

By applying inverse Laplace transform on (i), we get

$$x(t) = \frac{-3}{2}(t-3)^2 u(t-3) - \frac{(t-3)^3}{6}\cdot u(t-3)$$

2. $$X(s) = s\left[\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s}\right], \text{ROC}: \sigma > 0$$

$$= \frac{1}{s} - \frac{e^{-s}}{s} - e^{-2s};\ \sigma > 0 \quad \text{...(ii)}$$

We know that, $u(t) \rightleftharpoons \frac{1}{s},\ \sigma > 0$

Using the time-shifting property of Laplace Transform,

$$u(t-1) \rightleftharpoons \frac{e^{-s}}{s}, \sigma > 0$$

Also, $\delta(t-2) \rightleftharpoons e^{-2s}$

By applying inverse Laplace transform on (ii), we get

$$x(t) = u(t) - u(t-1) - \delta(t-2)$$

(ii) $$h(n) = \alpha^n u(n) \text{ with } |\alpha| < 1$$

$$H(z) = \frac{1}{1-\alpha z^{-1}}, \text{ ROC: } |z| > |\alpha|$$

Considering the unit step input i.e.

$$x(n) = u(n)$$

$\therefore$ $$X(z) = \frac{1}{1-z^{-1}}, \text{ ROC: } |z| > 1$$

We have, $$Y(z) = X(z)\cdot H(z)$$

$\therefore$ $$Y(z) = \frac{1}{1-z^{-1}}\cdot\frac{1}{1-\alpha z^{-1}}\text{, ROC: } |z|>1$$

$$= \frac{A}{1-z^{-1}}+\frac{B}{1-\alpha z^{-1}}$$

$\therefore$ $$Y(z) = \frac{\frac{1}{1-\alpha}}{1-z^{-1}}-\frac{\frac{\alpha}{1-\alpha}}{1-\alpha z^{-1}}$$

$\Rightarrow$ $$y(n) = \frac{1}{1-\alpha}u(n)-\frac{\alpha}{1-\alpha}(\alpha)^n u(n)$$

$$= \frac{1}{1-\alpha}u(n)-\frac{\alpha^{n+1}}{1-\alpha}u(n)$$

$\therefore$ $$y(n) = \frac{1-\alpha^{n+1}}{1-\alpha}u(n)$$

When $n\to\infty$,

$$\alpha^{n+1} = 0 \qquad (\because |\alpha| < 1)$$

$\therefore$ Step response of the system

$$y(n) = \frac{1}{1-\alpha}\text{ when } n\to\infty$$

Q.6 (c) Solution:

(i) Bilinear transformation relationship converting s-domain expression into an equivalent z-domain expression is given by,

$$s = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$$

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right] = \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right)\left(\frac{re^{-j\omega}+1}{re^{-j\omega}+1}\right)$$

$$= \frac{2}{T}\cdot\frac{r^2-1+r[e^{j\omega}-e^{-j\omega}]}{r^2+1+r[e^{j\omega}+e^{-j\omega}]}$$

$$= \frac{2}{T}\cdot\frac{r^2-1+j2r\sin\omega}{r^2+1+2r\cos\omega}$$

$\Rightarrow$ $$\sigma + j\Omega = \frac{2}{T}\cdot\frac{r^2-1}{r^2+1+2r\cos\omega}+j\frac{4}{T}\cdot\frac{r\sin\omega}{r^2+1+2r\cos\omega}$$

By comparison of LHS and RHS

$$\sigma = \frac{2}{T} \cdot \frac{r^2 - 1}{r^2 + 1 + 2r\cos\omega} \quad \text{...(a)}$$

and
$$\Omega = \frac{4}{T} \cdot \frac{r\sin\omega}{r^2 + 1 + 2r\cos\omega} \quad \text{...(b)}$$

From equation (a),

If $r = 1$, then $\sigma = 0$

If $r < 1$, then $\sigma < 0$

and if $r > 1$, then $\sigma > 0$

(ii) The Fourier series coefficient for a periodic signal $x[n]$ with period N is given by

$$a_k = \frac{1}{N} \sum_{<N>} x[n] \cdot e^{-jk\frac{2\pi}{N}n}$$

For the given signal $x(n)$,

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} x[n] \cdot e^{-jk\frac{2\pi}{N}n}$$

Letting $m = n + N_1$, the equation becomes

$$a_k = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)}$$

$$= \frac{1}{N} e^{jk(2\pi/N)N_1} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}$$

The summation consists of the sum of the first $2N_1 + 1$ terms in a geometric series, thus we get

$$a_k = \frac{1}{N} e^{jk(2\pi/N)N_1} \left(\frac{1 - e^{-jk2\pi(2N_1+1)/N}}{1 - e^{-jk(2\pi/N)}} \right)$$

$$= \frac{1}{N} \left(\frac{e^{-jk(2\pi/2N)} \left[e^{jk2\pi(2N_1+1/2)/N} - e^{-jk2\pi(N_1+1/2)/N} \right]}{e^{-jk(2\pi/2N)} \left[e^{jk(2\pi/2N)} - e^{-jk(2\pi/2N)} \right]} \right)$$

$$= \frac{1}{N} \frac{\sin\left[2\pi k\left(N_1 + \frac{1}{2} \right) / N \right]}{\sin\left(\frac{\pi k}{N} \right)}, \qquad k \neq 0, \pm N, \pm 2N \ldots$$

and
$$a_k = \frac{2N_1 + 1}{N}, \qquad k = 0, \pm N, \pm 2N, \ldots$$

Q.7 (a) Solution:

(i) 1. Given information are:

(a) $x(n)$ is right-sided.

(b) $X(z)$ has single pole.

(c) $x(0) = 4,\ x(2) = \frac{1}{4}$

From (b), $$X(z) = \frac{K}{1 - az^{-1}}$$

By applying inverse z-transform

$$x(n) = K(a)^n u(n) \quad \text{...(i)}$$

as $x(n)$ is right sided.

Put $n = 0$: $x(0) = K = 4$ ...from information (c)

$\Rightarrow$ $K = 4$

Put $n = 2$:

$$x(2) = K \cdot a^2 = \frac{1}{4}$$

$$a^2 = \frac{1}{4K} = \frac{1}{16}$$

$\Rightarrow$ $$a = \frac{1}{4}$$

From (i), $$x(n) = K(a)^n u(n)$$

$$= 4\left(\frac{1}{4}\right)^n u(n)$$

2. Given information are:

(a) Poles: $P_1 = \frac{1}{4}, P_2 = -1$

(b) ROC includes $|z| = \frac{1}{2}$

(c) $x(1) = 1, \quad x(-1) = 1$

From (a), $$X(z) = \frac{K}{(z - P_1)(z - P_2)}$$

$$= \frac{K}{\left(z-\frac{1}{4}\right)(z+1)}$$

$$= \left[\frac{K\cdot z^2}{\left(z-\frac{1}{4}\right)(z+1)}\right]z^{-2}$$

$$= \left[\frac{K}{\left(1-\frac{1}{4}z^{-1}\right)\left(1+z^{-1}\right)}\right]z^{-2}$$

$\Rightarrow$ $$X(z) = F(z)z^{-2} \quad \text{...(i)}$$

where, $$F(z) = \frac{K}{\left(1-\frac{1}{4}z^{-1}\right)(1+z^{-1})}$$

$$F(z) = K\left[\frac{A}{1-\frac{1}{4}z^{-1}} + \frac{B}{1+z^{-1}}\right]$$

Residue calculation:

$$A = \left(\frac{1}{1+z^{-1}}\right)\Bigg|_{z^{-1}=4} = \frac{1}{1+4} = \frac{1}{5}$$

$$B = \left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)\Bigg|_{z^{-1}=-1} = \frac{1}{1+\frac{1}{4}} = \frac{4}{5}$$

Thus, $$F(z) = K\left[\frac{1}{5}\cdot\frac{1}{1-\frac{1}{4}z^{-1}} + \frac{4}{5}\cdot\frac{1}{1+z^{-1}}\right] \quad \text{...(ii)}$$

From information (b), since ROC includes $|z| = \frac{1}{2}$. Thus, ROC: $\frac{1}{4} < |z| < 1$

i.e., $x(n)$ will be a both sided signal.

By applying inverse z-transform on (ii)

$$F(z) = K\left[\frac{1}{5}\cdot\left(\frac{1}{4}\right)^n u(n) - \frac{4}{5}\cdot(-1)^n u(-n-1)\right]$$

From (i), $$X(z) = F(z)z^{-2}$$

By using time shifting property of z-transform,

$$x(n) = f(n-2)$$

$$\Rightarrow \quad x(n) = K\left[\frac{1}{5}\left(\frac{1}{4}\right)^{n-2} u(n-2) - \frac{4}{5}(-1)^{n-2} u(-(n-2)-1)\right]$$

$$x(n) = K\left[\frac{1}{5}\left(\frac{1}{4}\right)^{n-2} u(n-2) - \frac{4}{5}(-1)^{n-2} u(-n+1)\right]$$

Put $n = 1$: $$x(1) = K\left[\frac{-4}{5}(-1)^{-1}\right] = 1 \quad \text{.....from information (c)}$$

$$\Rightarrow \quad K = \frac{5}{4}$$

Using the above value of K and substituting $n = -1$, we can verify,

$$x(-1) = K\left[\frac{-4}{5}(-1)^{-3}\right] = 1 \quad \text{...from information (c)}$$

Thus, $$x(n) = \frac{5}{4}\left[\frac{1}{5}\left(\frac{1}{4}\right)^{n-2} u(n-2) - \frac{4}{5}(-1)^{n-2} u(-n+1)\right]$$

$$\Rightarrow \quad x(n) = \left(\frac{1}{4}\right)^{n-1} u(n-2) - (-1)^{n-2} u(-n+1)$$

(ii) 1. The complex Fourier series coefficient is given by

$$C_n = \frac{1}{T}\int_0^T f(t) e^{-jn\omega_0 t} dt$$

The given signal is periodic with time period $T = 1$ and $f(t) = t$ for $0 \le t \le 1$

$$= \frac{1}{1}\int_0^1 t e^{-jn\cdot\frac{2\pi}{1}t} dt = \left[t \times \frac{e^{-j2\pi nt}}{-j2\pi n}\right]_0^1 - \int_0^1 \frac{e^{-j2\pi nt}}{-j2\pi n} dt$$

Now, $$e^{-jn\pi} = (-1)^n$$

$\therefore$ $$e^{-j2n\pi} = (-1)^2 = 1$$

$\therefore$ $$C_n = \frac{1}{-j2\pi n} - \left[\frac{e^{-j2\pi nt}}{(-j2\pi n)^2}\right]_0^1 = \frac{1}{-j2\pi n} + \frac{1}{(2\pi n)^2} - \frac{1}{(2\pi n)^2}$$

$$C_n = \frac{j}{2\pi n}$$

but C_n becomes ∞ when we put $n = 0$, so again calculate C_n at $n = 0$

We have, $C_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}dt$

putting, $n = 0$

$\therefore$ $C_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{1}\int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1$

$\therefore$ $C_0 = \frac{1}{2}$

$$\therefore \quad C_n = \begin{cases} \frac{1}{2} & \text{when } n = 0 \\ \frac{j}{2\pi n} & \text{when } n \neq 0 \end{cases}$$

Thus, the complex Fourier series representation of a signal is given by

$$x(n) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt}$$

2. Magnitude, $C_0 = \frac{1}{2}$

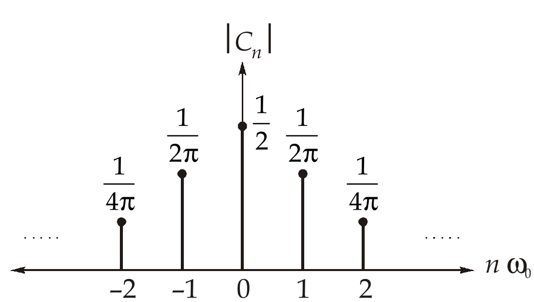
and $|C_n| = \left|\frac{j}{2\pi n}\right| = \frac{1}{2\pi n}$

Phase, $\angle C_n = +j = \frac{\pi}{2}$ when $n > 0$

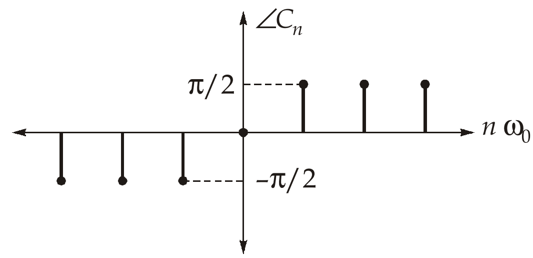
$= -j = \frac{-\pi}{2}$ when $n < 0$

$= 0$ when $n = 0$

Magnitude plot:



Phase plot:



Q.7 (b) Solution:

(i) From the given information, we have the Fourier transform $G(e^{j\omega})$ of $g[n]$ to be

$$G(e^{j\omega}) = g[0] + g[1]\,e^{-j\omega},$$

Also, when the input to the system is $x[n] = \left(\frac{1}{4}\right)^n u[n]$, the output is $g[n]$. Therefore

$$H(e^{j\omega}) = \frac{G(e^{j\omega})}{X(e^{j\omega})}$$

We obtain,
$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

Therefore,
$$H(e^{j\omega}) = \left[\left(g(0)+g(1)e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)\right]$$
$$= \left[g(0)+\left[g(1)-\frac{1}{4}g(0)\right]e^{-j\omega}-\frac{g(1)}{4}e^{-2j\omega}\right]$$

Clearly, $h[n]$ is a three point sequence.

We have,
$$H(e^{j\omega}) = h(0) + h(1)\,e^{-j\omega} + h(2)\,e^{-2j\omega}$$

and
$$H(e^{j(\omega-\pi)}) = h(0) + h(1)\,e^{-j(\omega-\pi)} + h(2)\,e^{-2j(\omega-\pi)}$$
$$= h(0) - h(1)\,e^{-j\omega} + h(2)\,e^{-2j\omega}$$

We see that,
$$H(e^{j\omega}) = H(e^{j(\omega-\pi)}) \text{ only if } h(1) = 0$$

We also have,
$$H(e^{j\pi/2}) = h(0) + h(1)\,e^{-j\pi/2} + h(2)\,e^{-2j\pi/2}$$
$$= h(0) - h(2) - jh(1)$$

Since we are also given that

$$\text{Re}\{H(e^{j\pi/2})\} = 1, \text{ we have}$$

$$h(0) - h(2) = 1 \qquad \text{...(i)}$$

Now note that,
$$g[n] = h[n]*\left[\left(\frac{1}{4}\right)^n u[n]\right] = \sum_{k=0}^{2} h[k]\left(\frac{1}{4}\right)^{n-k} u[n-k]$$

Evaluating this equation at $n = 2$, we have

$$g(2) = 0 = \frac{1}{16}h[0] + \frac{1}{4}h[1] + h[2]$$

Since $h[1] = 0, \frac{1}{16}h[0] + h[2] = 0$ ...(ii)

Solving equations (i) and (ii), we obtain

$$h[0] = \frac{16}{17} \text{ and } h[2] = \frac{-1}{17}$$

Therefore, $$h[n] = \frac{16}{17}\delta[n] - \frac{1}{17}\delta[n-2].$$

(ii) 1. Properties of ROC of Laplace transform:

(a) For rational Laplace transform, ROC does not contain any pole.

(b) If the signal is of finite duration and absolutely integrable, then ROC is entire s-plane.

(c) ROC consists of strips parallel to $j\omega$-axis in s-plane.

(d) If signal is right sided and $\text{Re}\{s\} = \sigma_0$ is in the ROC, then all the values of s for which $\text{Re}\{s\} > \sigma_0$ will also be in the ROC.

(e) If signal is left hand sided and $\text{Re}\{s\} = \sigma_0$ is in the ROC, then all the values of s for which $\text{Re}\{s\} < \sigma_0$ will also be in the ROC.

(f) If $x(t)$ is both sided signal, then ROC will be a strip in the s-plane.

(g) If $x(t)$ is rational, then ROC will be bounded by poles or extends to infinity.

(h) For a stable system, ROC includes the $j\omega$-axis.

2.

$$H(s) = \ln\left(\frac{1}{3s+2}\right)$$

$$H(s) = -\ln(3s+2)$$

Differentiating w.r.t s, we get

$$\frac{d}{ds}H(s) = -\frac{1}{3s+2} \times 3$$

$$\therefore \quad \frac{d}{ds}H(s) = \frac{-3}{3s+2}$$

From the multiplication by 't' property of Laplace Transform,

$$x(t) \xleftrightarrow{LT} X(s)$$

then $$-t\,x(t) \xleftrightarrow{LT} \frac{d}{ds}X(s)$$

We have,

$$-t\,h(t) \xleftrightarrow{LT} \frac{-3}{3s+2}$$

$$\therefore \quad -t\,h(t) = ILT\left[\frac{-3}{3s+2}\right] = ILT\left[\frac{-1}{s+2/3}\right]$$

$$-t\,h(t) = -e^{-2/3t}u(t)$$

$$\therefore \quad h(t) = \frac{e^{-2/3t}}{t}u(t)$$

Q.7 (c) Solution:

(i) **1.** We know that $$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}nk},$$

$$k = 0, 1.., N-1$$

$$\therefore \quad X(0) = \sum_{n=0}^{5} x(n)e^{0} = \sum_{n=0}^{5} x(n) = 1-2+3+0-1+1 = 2$$

2. $$X(3) = \sum_{n=0}^{5} x(n)\,e^{-j\frac{2\pi}{6}3n} = \sum_{n=0}^{5} x(n)\,(-1)^n$$

$$= x(0)\,(1) + x(1)\,(-1) + x(2)\,(1) + x(3)\,(-1) + x(4)\,(1) + x(5)\,(-1)$$

$$= (1)\,(1) - 2(-1) + 3(1) + (0)\,(-1) - 1(1) + 1(-1) = 4$$

3. We have, $$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,e^{j\frac{2\pi}{6}nk},\ n = 0, 1, ..., N-1$$

$$\therefore \quad \sum_{k=0}^{N-1} X(k) = Nx(0) = 6(1) = 6$$

4. From Parseval's theorem,

$$\sum_{k=0}^{5}|X(k)|^2 = N\sum_{n=0}^{5}|x(n)|^2 = 6\left[(1)^2+(-2)^2+(3)^2+(0)^2+(-1)^2+(1)^2\right]$$

$$= 6(1 + 4 + 9 + 0 + 1 + 1) = 96$$

(ii) Since $X(s)$ has 4 poles and no zeros in the finite s-plane, we may assume that $X(s)$ is of the form,

$$X(s) = \frac{A}{(s-a)(s-b)(s-c)(s-d)}$$

Since $x(t)$ is real, the poles of $X(s)$ must occur in conjugate reciprocal pairs.

Therefore, we may assume that $b = a^*$ and $d = c^*$.

$$X(s) = \frac{A}{(s-a)(s-a^*)(s-c)(s-c^*)}$$

Since, $x(t)$ is even, $X(s)$ must also be even. Hence, the poles must be symmetric about the $j\Omega$-axis. Therefore, $c = -a^*$

Therefore, $$X(s) = \frac{A}{(s-a)(s-a^*)(s+a^*)(s+a)}$$

It is given that the location of one of the poles is $\left(\frac{1}{2}\right)e^{j\frac{\pi}{4}}$. If we assume that this pole is a, we have

$$X(s) = \frac{A}{\left(s-\frac{1}{2}e^{j\frac{\pi}{4}}\right)\left(s-\frac{1}{2}e^{-j\frac{\pi}{4}}\right)\left(s+\frac{1}{2}e^{-j\frac{\pi}{4}}\right)\left(s+\frac{1}{2}e^{j\frac{\pi}{4}}\right)}$$

$$X(s) = \frac{A}{\left(s^2-\frac{s}{\sqrt{2}}+\frac{1}{4}\right)\left(s^2+\frac{s}{\sqrt{2}}+\frac{1}{4}\right)}$$

Also, we are given that

$$\int_{-\infty}^{\infty} x(t)dt = X(0) = 4$$

Substituting in the above expression for $X(s)$, we have $A = \frac{1}{4}$.

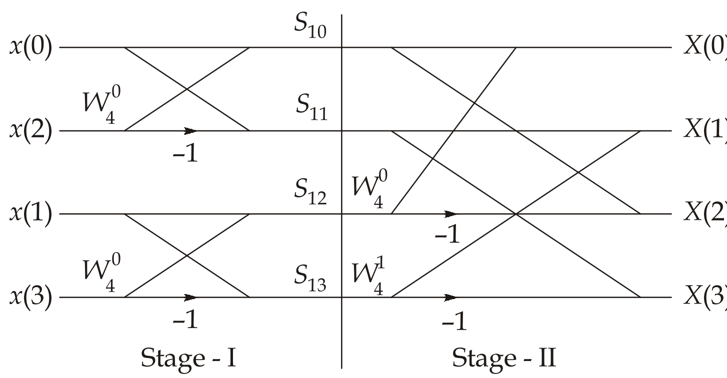
Therefore, $$X(s) = \frac{\left(\frac{1}{4}\right)}{\left(s^2-\frac{s}{\sqrt{2}}+\frac{1}{4}\right)\left(s^2+\frac{s}{\sqrt{2}}+\frac{1}{4}\right)}$$

Q.8 (a) Solution:

To use a Radix-2 DIT FFT, No. of samples in $x(n)$ and $h(n)$ must be equal and multiple of 2, so we add the required number of zero's to the $x(n)$ and $h(n)$.

$$\therefore \quad x(n) = \{2, 2, 4, 0\}$$
$$\uparrow$$

$$\text{and} \quad h(n) = \{1, 1, 0, 0\}$$



We have, $W_N^k = e^{-jk\frac{2\pi}{N}}$. Thus, $W_4^0 = 1$ and $W_4^1 = e^{-j\frac{\pi}{2}} = -j$

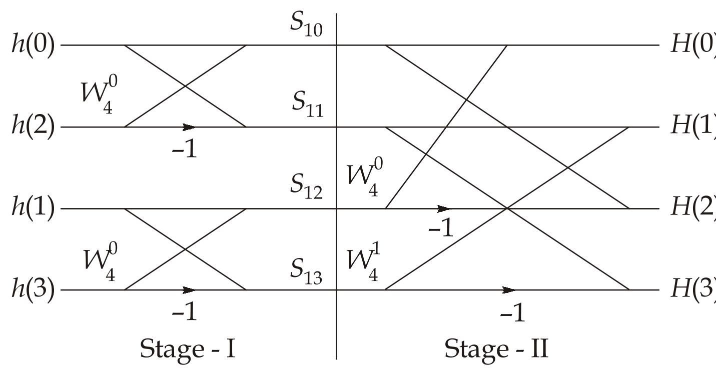
Output of stage-I:

$$S_{10} = x(0) + x(2) = 6$$
$$S_{11} = x(0) - x(2) = -2$$
$$S_{12} = x(1) + x(3) = 2$$
$$S_{13} = x(1) - x(3) = 2$$

Output of stage-II:

$$X(0) = S_{10} + S_{12} = 8$$
$$X(1) = S_{11} - jS_{13} = -2 - 2j$$
$$X(2) = S_{10} - S_{12} = 4$$
$$X(3) = S_{11} + jS_{13} = -2 + 2j$$
$$\therefore \quad X(k) = \{8, -2 - 2j, 4, -2 + 2j\}$$

Similarly, lets find $H(k)$



Output of stage-I

$$S_{10} = h(0) + h(2) = 1$$
$$S_{11} = h(0) - h(2) = 1$$
$$S_{12} = h(1) + h(3) = 1$$
$$S_{13} = h(1) - h(3) = 1$$

Output of stage-II

$$H(0) = S_{10} + S_{12} = 2$$
$$H(1) = S_{11} - jS_{13} = 1 - j$$
$$H(2) = S_{10} - S_{12} = 0$$
$$H(3) = S_{11} + jS_{13} = 1 + j$$

$\therefore$ $$H(k) = \{2, 1 - j, 0, 1 + j\}$$

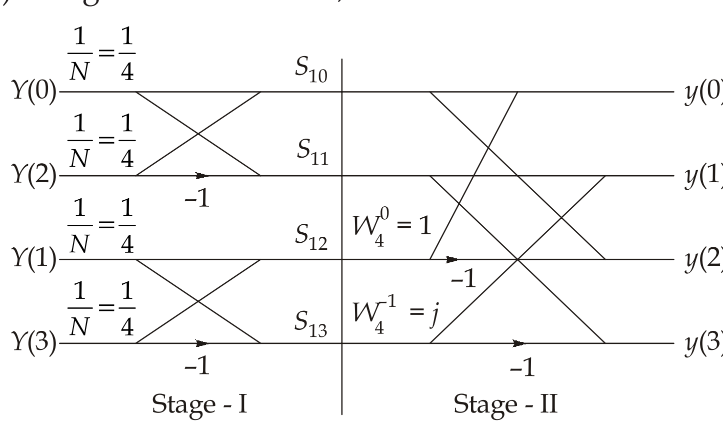
We know that,

$$y(n) = x(n) * h(n)$$

Therefore, $$Y(k) = X(k) \cdot H(k)$$
$$= \{8, -2 - 2j, 4, -2 + 2j\} \cdot \{2, 1 - j, 0, 1 + j\}$$

$\therefore$ $$Y(k) = \{16, -4, 0, -4\}$$

Now lets find $y(n)$ using inverse DIT FFT,



Output of stage-I

$$S_{10} = \frac{1}{4}[Y(0)+Y(2)] = \frac{1}{4}\times 16 = 4$$

$$S_{11} = \frac{1}{4}[Y(0)-Y(2)] = \frac{1}{4}\times 16 = 4$$

$$S_{12} = \frac{1}{4}[Y(1)+Y(3)] = \frac{1}{4}\times -8 = -2$$

$$S_{13} = \frac{1}{4}[Y(1)-Y(3)] = \frac{1}{4}\times 0 = 0$$

Output of stage-II

$$y(0) = S_{10} + S_{12} = 2$$

$$y(1) = S_{11} + jS_{13} = 4$$

$$y(2) = S_{10} - S_{12} = 6$$

$$y(3) = S_{11} - jS_{13} = 4$$

$$\therefore \quad y(n) = \{\underset{\uparrow}{2}, 4, 6, 4\}$$

Q.8 (b) Solution:

(i) The zero input response is the response due to initial conditions only. So we make input $x(n) = 0$. Now the difference equation becomes

$$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2) = 0$$

Taking z-transform on both sides, we have

$$Y(z)-\frac{3}{4}[z^{-1}Y(z)+y(-1)]+\frac{1}{8}[z^{-2}\,Y(z)+z^{-1}\,y(-1)+y(-2)] = 0$$

i.e. $$Y(z)\left(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right)-\frac{1}{8} = 0$$

$$\therefore \quad Y(z) = \frac{1/8}{1-(3/4)z^{-1}+(1/8)z^{-2}} = \frac{(1/8)z^2}{z^2-(3/4)z+(1/8)}$$

$$= \frac{(1/8)z^2}{[z-(1/2)][z-(1/4)]}$$

The partial fraction expansion of $Y(z)/z$ gives

$$\frac{Y(z)}{z} = \frac{(1/8)z}{\left[z-\left(\frac{1}{2}\right)\right]\left[z-\left(\frac{1}{4}\right)\right]} = \frac{A}{z-\left(\frac{1}{2}\right)} + \frac{B}{z-\left(\frac{1}{4}\right)} = \frac{+\frac{1}{4}}{z-\left(\frac{1}{2}\right)} - \frac{1/8}{z-\frac{1}{4}}$$

$$\therefore \quad y(n) = \frac{1}{4}\left(\frac{1}{2}\right)^n u(n) - \frac{1}{8}\left(\frac{1}{4}\right)^n u(n)$$

(ii) To find the zero state response due to a step input, $x(n) = u(n)$. Thus, we have

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = u(n) + u(n-1)$$

We know that the zero state response is due to the input alone. So for zero state response, the initial conditions are neglected. Taking Z-transform on both sides of the above equation and neglecting the initial conditions, we have

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = U(z) + z^{-1}U(z) = \frac{z}{z-1} + \frac{1}{z-1}$$

i.e. $$Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) = \frac{z+1}{z-1}$$

$$\therefore \quad Y(z) = \frac{z+1}{(z-1)[1-(3/4)z^{-1}+(1/8)z^{-2}]} = \frac{z^2(z+1)}{(z-1)[z^2-(3/4)z+(1/8)]}$$

$$= \frac{z^2(z+1)}{(z-1)[z-(1/2)][z-(1/4)]}$$

Taking partial fraction of $Y(z)/z$, we have

$$\therefore \quad \frac{Y(z)}{z} = \frac{z(z+1)}{(z-1)[z-(1/2)][z-(1/4)]} = \frac{A}{z-1} + \frac{B}{z-(1/2)} + \frac{C}{z-(1/4)}$$

$$= \frac{16/3}{z-1} - \frac{6}{z-(1/2)} + \frac{5/3}{z-(1/4)}$$

or $$Y(z) = \frac{16}{3}\left(\frac{z}{z-1}\right) - 6\left[\frac{z}{z-(1/2)}\right] + \frac{5}{3}\left[\frac{z}{z-(1/4)}\right]$$

Taking the inverse z-transform on both sides, we get zero state response for a step input as,

$$y(n) = \frac{16}{3}u(n) - 6\left(\frac{1}{2}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{4}\right)^n u(n)$$

Q.8 (c) Solution:

Direct form-I

Given, $$y(n) = -\frac{13}{12}y(n-1)-\frac{9}{24}y(n-2)-\frac{1}{24}y(n-3)+x(n)+4x(n-1)+3x(n-2)$$

Taking Z-transform on both sides, we get

$$Y(z) = -\frac{13}{12}z^{-1}Y(z)-\frac{9}{24}z^{-2}Y(z)-\frac{1}{24}z^{-3}Y(z)+X(z)+4z^{-1}X(z)+3z^{-2}X(z)$$

The direct form-I structure can be obtained from the above equation as shown in Figure (1).

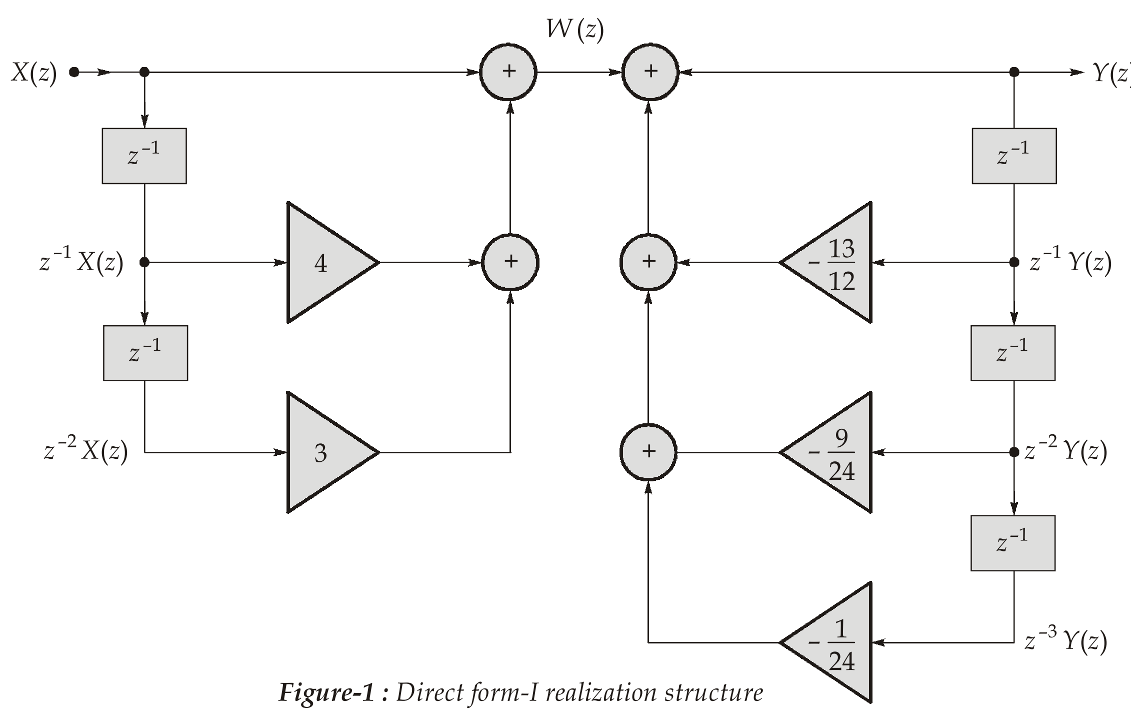


Figure-1 : *Direct form-I realization structure*

Direct form-II

Taking Z-transform of the given difference equation, we have

$$Y(z) = -\frac{13}{12}z^{-1}Y(z)-\frac{9}{24}z^{-2}Y(z)-\frac{1}{24}z^{-3}Y(z)+X(z)+4z^{-1}X(z)+3z^{-2}X(z)$$

i.e. $$Y(z)+\frac{13}{12}z^{-1}Y(z)+\frac{9}{24}z^{-2}Y(z)+\frac{1}{24}z^{-3}Y(z) = X(z) + 4z^{-1}X(z) + 3z^{-2}X(z)$$

i.e. $$Y(z)\left[1+\frac{13}{12}z^{-1}+\frac{9}{24}z^{-2}+\frac{1}{24}z^{-3}\right] = X(z)\,[1 + 4z^{-1} + 3z^{-2}]$$

Therefore, the transfer function of the system is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+4z^{-1}+3z^{-2}}{1+\left(\frac{13}{12}\right)z^{-1}+\left(\frac{9}{24}\right)z^{-2}+\left(\frac{1}{24}\right)z^{-3}}$$

Let $$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

where, $$\frac{W(z)}{X(z)} = \frac{1}{1+\left(\frac{13}{12}\right)z^{-1}+\left(\frac{9}{24}\right)z^{-2}+\left(\frac{1}{24}\right)z^{-3}}$$

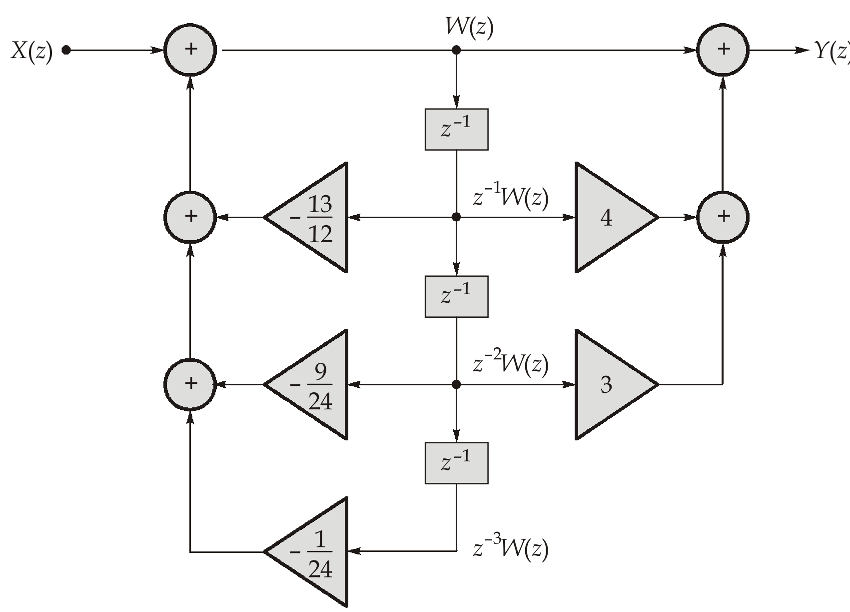


***Figure-2 :** Direct form-II realization structure*

and $$\frac{Y(z)}{W(z)} = 1 + 4z^{-1} + 3z^{-2}$$

On cross multiplying the above equations, we get

$$W(z)\left[1+\frac{13}{12}z^{-1}+\frac{9}{24}z^{-2}+\frac{1}{24}z^{-3}\right] = X(z)$$

i.e. $$W(z) = X(z)-\frac{13}{12}z^{-1}W(z)-\frac{9}{24}z^{-2}\,W(z)-\frac{1}{24}z^{-3}\,W(z)$$

and $$Y(z) = W(z) + 4z^{-1}\,W(z) + 3z^{-2}\,W(z)$$

The above equations for $W(z)$ and $Y(z)$ can be realized by a direct form-II structure as shown in figure (2).

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