Detailed Solutions

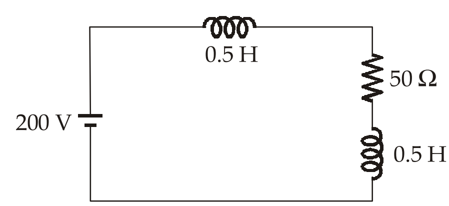
ESE-2026 Mains Test Series

Electrical Engineering Test No : 1

Section A : Electrical Circuits

Q.1 (a) Solution:

For current at t = 10 ms after switch is closed, the circuit will become,



$$i(t) = i_\infty \cdot \left[1 - e^{-\frac{t}{\tau}}\right]$$

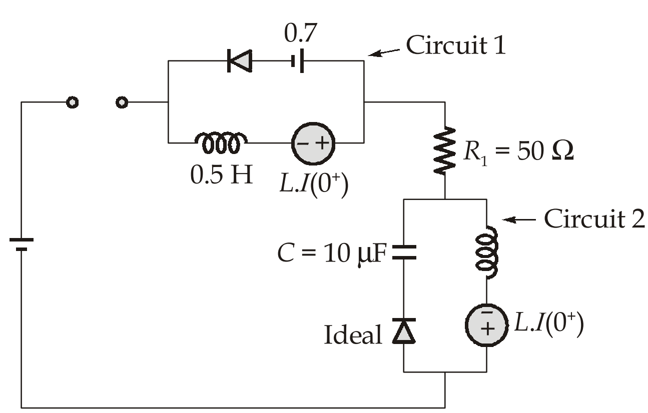
$$i_\infty = \frac{200}{50} = 4 \text{ A},$$

$$\tau = \frac{L}{R} = \frac{0.5 + 0.5}{50} = \frac{1}{50}$$

$\therefore$ $$i(t) = 4[1 - e^{-50t}]$$

At 10 ms, $$I_0 = i(10 \text{ ms}) = 4[1 - e^{-50 \times 10 \text{ ms}}] = 1.57 \text{ A}$$

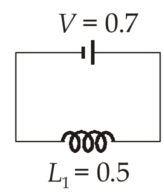
After opening the switch circuit will become,



Both the diodes will be forward biased and current will not flow in the resistance hence current in $R_1 = 0$ A.

In circuit 1:

$$L\frac{di}{dt} + V = 0$$



$$\int_{I_0}^{I} di = \int_0^{8\text{ms}} -\frac{V}{L} dt$$

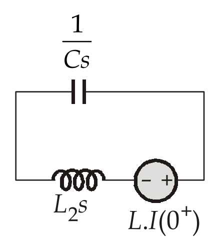
$$I - I_0 = -\frac{V}{L} \cdot (t)_0^{8\text{ ms}}$$

$$I = I_0 - \frac{V}{L} \times 8ms$$

Current in inductor L_1

$\Rightarrow$ $$I = 1.57 - \left[\frac{0.7}{0.5} \times 8 \text{ ms}\right] = 1.558 \text{ A}$$

In circuit 2:



$$-L\,I(0^+) + I(s)\frac{1}{Cs} + Ls\,I(s) = 0$$

$$I(s) = \frac{L\,I(0^+)}{\frac{1}{Cs} + Ls}$$

Solving we get, $$I(s) = I(0) \cdot \frac{s}{s^2 + \frac{1}{LC}}$$ taking Inverse Laplace Transform

$\therefore$ $$i(t) = I(0) \cdot \cos(\omega_0 t)$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.5 \times 10 \times 10^{-6}}} = 447.21 \text{ rad/s}$$

At, $$t = 8 \text{ ms}$$

$$\omega_0 t = 204.98°$$

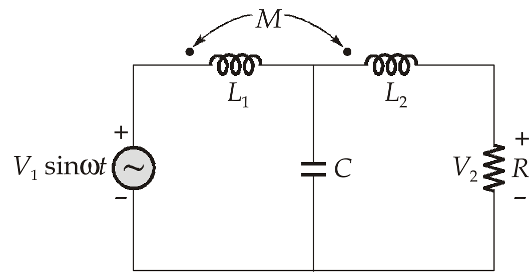
Current in inductor L_2, $$i(t) = 1.57 \times \cos(204.9) = -1.42 \text{ A}$$

Voltage across C_1:

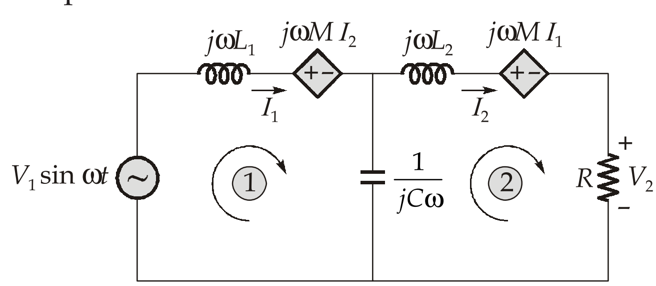
$$V = \frac{1}{C}\int i(t)\,dt = \frac{1}{C}\int_0^{8ms} I_0 \cdot \cos(\omega_0 t)\,dt = \frac{I_0}{C\omega_0}\sin\omega_0 t$$

$$= \frac{1.57}{4.4721 \times 10^{-3}} \times \sin(204.9) = -147.81 \text{ V}$$

Q.1 (b) Solution:



Using decoupled technique:



Applying KVL in loop (1): $$V_1 = j\omega L_1 I_1 + j\omega M I_2 + \frac{1}{j\omega C}(I_1 - I_2)$$

In loop (2) KVL gives,

$$+V_2 + j\omega L_2 I_2 + j\omega M I_1 + \frac{1}{j\omega C}(I_2 - I_1) = 0$$

Taking in Laplace domain, $$V_1(s) = L_1 sI_1 + sMI_2 + \frac{1}{Cs}(I_1 - I_2) \quad ...(i)$$

$$V_2(s) = -sL_2I_2 - sMI_1 - \frac{1}{Cs}(I_2 - I_1) \quad ...(ii)$$

Also, $$V_2(s) = I_2 \cdot R$$

Putting this in (ii), $$I_2 R = -sLI_2 - sMI_1 + \frac{1}{Cs}[I_2 - I_1]$$

$$I_2\left[R + sL + \frac{1}{Cs}\right] = I_1\left[\frac{1}{Cs} - sM\right]$$

$$I_2 = I_1 \cdot \left[\frac{\frac{1}{Cs} - sM}{R + sL + \frac{1}{Cs}}\right] \quad ...(iii)$$

From equation (i), $$V_1 = \left(Ls + \frac{1}{Cs}\right)I_1 + I_2\left(sM - \frac{1}{Cs}\right) \quad ...(iv)$$

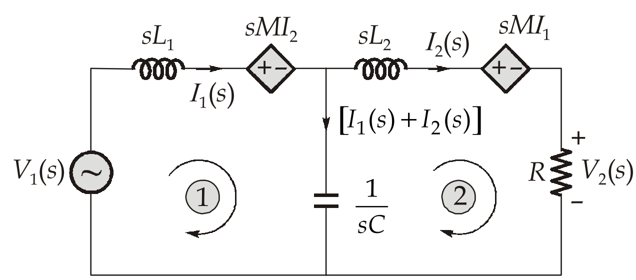
Putting (iii) in (iv),
$$V_1 = \left(Ls + \frac{1}{Cs}\right) \times \frac{\left(R + sL + \frac{1}{Cs}\right) \times I_2}{\frac{1}{Cs} - sM} + I_2\left(sM - \frac{1}{Cs}\right)$$

$$V_2 = I_2 \cdot R$$

$$\frac{V_1}{V_2} = \frac{\left(Ls + \frac{1}{Cs}\right)\left(R - sL - \frac{1}{Cs}\right)}{\left(\frac{1}{Cs} - sM\right)R} + \left(sM - \frac{1}{Cs}\right) \cdot \frac{1}{R}$$

Alternate Solution:

Circuit in s-domain



Bty KVL in loop-1,

$$V(s) = sL_1 I_1(s) + sMI_2(s) + \frac{1}{sC}[I_1(s) - I_2(s)]$$

$$V(s) = \left[sL_1 + \frac{1}{sC}\right]I_1(s) + \left[sM - \frac{1}{sC}\right]I_2(s) \qquad \text{...(i)}$$

KVL in loop-2,

$$\left[sL_2 + R + \frac{1}{sC}\right]I_2(s) = \left[\frac{1}{sC} - \frac{1}{sC}\right]I_1(s) \qquad \text{...(ii)}$$

Put equation (ii) in equation (i),

$$V(s) = \left[\left[sL_1 + \frac{1}{sC}\right] \cdot \frac{\left[s^2L_2C + 1 - sRC\right]}{\left[1 - s^2MC\right]} + \frac{\left[s^2MC - 1\right]}{sC}\right]I_2(s)$$

$$V(s) = \left[\frac{(s^2L_1C + 1)\left[s^2L_2C + sRC + 1\right]}{sRC\left[1 - s^2MC\right]} + \frac{\left[s^2MC - 1\right]}{sRC}\right]V_2(s)$$

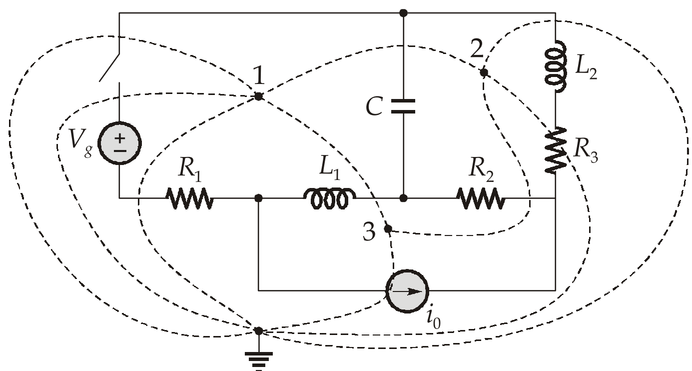
$$V_1(s) = \frac{(s^2L_1C + 1)(s^2L_2C + sRC + 1) - (s^2MC - 1)^2}{sRC(1 - s^2MC)}V_2(s)$$

Therefore transfer function:

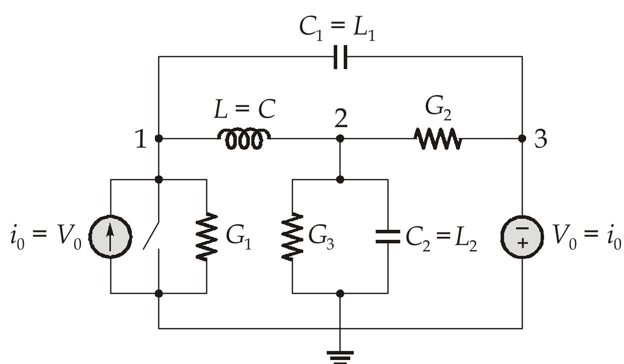
$$\frac{V_2(s)}{V_1(s)} = \frac{sRC(1-s^2MC)}{(s^2L_1C+1)(s^2L_2C+SRC(+1)-(s^2MC-1)^2}$$

Q.1 (c) Solution:

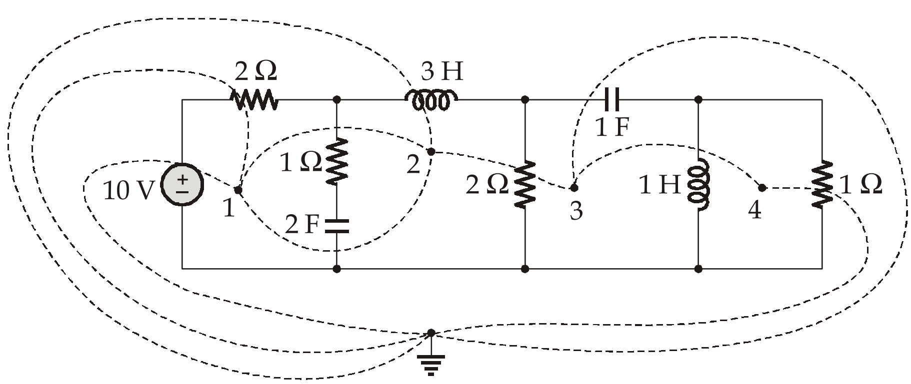
1. The dual network is drawn as shown below,



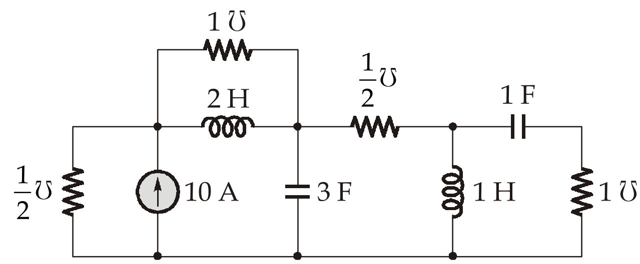
The final dual circuit becomes as shown below,



2. The dual network is drawn as shown below,



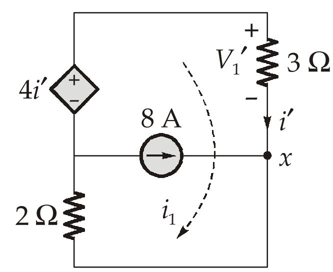
The final dual network is shown below,



Q.1 (d) Solution:

Case-I: When the 8-A current source is acting alone

By KVL for the supermesh,



$$3i + 2i_1 - 4i' = 0$$

$$\Rightarrow \quad i_1 = \frac{1}{2}i'$$

By KCL at node x,

$$i_1 = (8 + i')$$

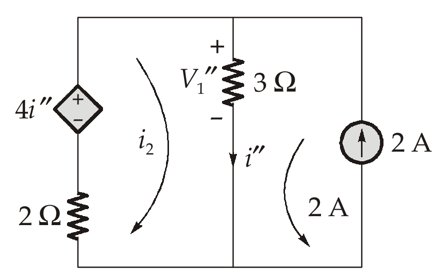
$$\Rightarrow \quad \frac{1}{2}i' = 8 + i'$$

$$\Rightarrow \quad i' = -16 \text{ A}$$

$$\therefore \quad V_1' = 3i' = 3 \times (-16) = -48 \text{ V}$$

Case-II: When the 2-A current source is acting alone

By KVL,



$$3(i_2 + 2) + 2i_2 - 4i'' = 0$$

$$\Rightarrow \quad 5i_2 + 6 - 4i'' = 0$$

Now, $$i'' = (i_2 + 2)$$

$$\therefore \quad 5i_2 + 6 - 4(i_2 + 2) = 0$$

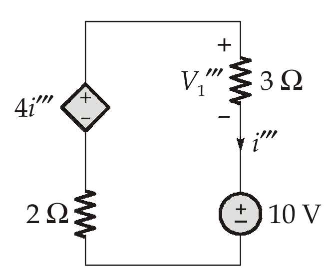
$$\Rightarrow \quad i_2 = 2 \text{ A}$$

$$\therefore \quad i'' = (i_2 + 2) = (2 + 2) = 4 \text{ A}$$

$$\therefore \quad V_1'' = 3i'' = 3 \times 4 = 12 \text{ V}$$

Case-III: When the 10-V current source is acting alone

By KVL,



$$3i''' - 10 + 2i''' - 4i''' = 0$$

$$\Rightarrow \quad i''' = 10 \text{ A}$$

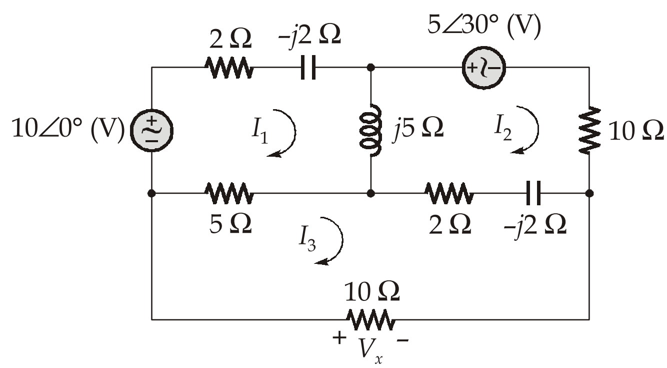
$$\therefore \quad V_1''' = 10 \times 3 = 30 \text{ V}$$

When all the sources are acting simultaneously, by the superposition theorem the voltage is given as

$$\begin{aligned} V_1 &= (V_1' + V_1'' + V_1''') \\ &= (-48 + 12 + 30) = -6 \text{ V} \end{aligned}$$

Q.1 (e) Solution:

By KVL for the three meshes, we get



$$(7 + j3)I_1 - j5I_2 - 5I_3 = 10 \quad \text{...(i)}$$

$$-j5I_1 + (12 + j3)5I_2 - (2 - j2)I_3 = -(4.33 + j2.5)$$

$$-5I_1 - (2 - j2)I_2 + (17 - j2)I_3 = 0$$

Solving for I_3 from equations (i), (ii), we get

$$I_3 = \frac{\begin{vmatrix} (7+j3) & -j5 & 10 \\ -j5 & (12+j3) & -(4.33+j2.5) \\ -5 & -(2-j2) & 0 \end{vmatrix}}{\begin{vmatrix} (7+j3) & -j5 & -5 \\ -j5 & (12+j3) & -(2-j2) \\ -5 & -(2-j2) & (17-j2) \end{vmatrix}}$$

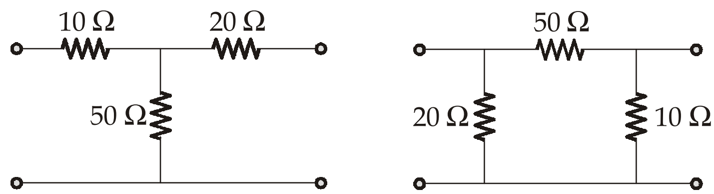
$$= 0.435\angle{-194.15°} \text{ (A)}$$

Therefore, the required voltage is

$$\begin{aligned} V_x &= 10 \times I_3 = 10 \times 0.435\angle{-194.15°} \\ &= 4.35\angle{-194.15°} \text{ (V)} \end{aligned}$$

Q.2 (a) (i) Solution:

This two-port network can be considered as the cascade connection of two two-port network as show below,



Network (a) Network (b)

For the network (a), as this is a T-network, the z-parameter are given as

$$z_{11} = 60\ \Omega; \qquad z_{12} = 50\ \Omega;$$

$$z_{22} = 70\ \Omega;$$

$$\therefore \quad \Delta z = (z_{11}z_{22} - z_{12}z_{21}) = (60 \times 70 - 50^2) = 1700$$

$$\therefore \quad A_a = \frac{z_{11}}{z_{21}} = \frac{60}{50} = \frac{6}{5}$$

$$B_a = \frac{\Delta z}{z_{21}} = \frac{1700}{50} = 34\ \Omega$$

$$C_a = \frac{1}{z_{21}} = \frac{1}{50}\text{mho}$$

$$D_a = \frac{z_{22}}{z_{21}} = \frac{70}{50} = \frac{7}{5}$$

For the network (b), as this is a π-network, the y-parameters are given as

$$y_{11} = \left(\frac{1}{50} + \frac{1}{20}\right) = \frac{7}{100}\text{mho};$$

$$y_{12} = y_{21} = -\frac{1}{50}\text{mho};$$

$$y_{22} = \left(\frac{1}{50} + \frac{1}{10}\right) = \frac{3}{25}\text{mho}$$

$$\therefore \quad \Delta y = (y_{11}y_{23} - y_{12}y_{21}) = \frac{7}{100} \times \frac{3}{25} - \left(-\frac{1}{50}\right)^2 = \frac{1}{125}$$

$$\therefore \quad A_b = -\frac{y_{22}}{y_{21}} = -\frac{3/25}{-1/50} = 6$$

$$B_b = -\frac{1}{y_{21}} = -\frac{1}{-1/50} = 50\,\Omega$$

$$C_b = -\frac{\Delta y}{y_{21}} = -\frac{1/125}{-1/50} = \frac{2}{5}\text{mho}$$

$$D_b = -\frac{y_{11}}{y_{21}} = -\frac{7/100}{-1/50} = \frac{7}{2}$$

For the entire network, the ABCD parameters are given as

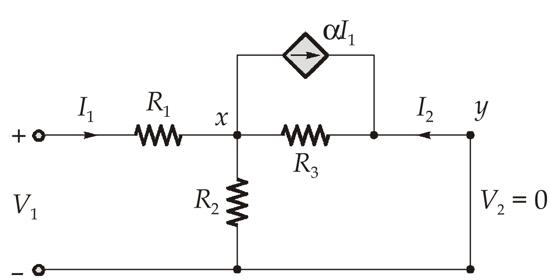
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \times \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{5} & 34 \\ \frac{1}{50} & \frac{7}{5} \end{bmatrix} \times \begin{bmatrix} 6 & 50 \\ \frac{2}{5} & \frac{7}{2} \end{bmatrix} = \begin{bmatrix} 20.8 & 179 \\ 0.68 & 5.9 \end{bmatrix}$$

Q.2 (a) (ii) Solution:

Case-I: When $V_2 = 0$

The circuit is modified as shown in figure,



By KCL at node x,

$$\frac{V_x}{R_2} + \frac{V_x}{R_3} + \alpha I_1 = I_1$$

$$\Rightarrow \quad V_x = (I - \alpha)\frac{R_2 R_3}{R_2 + R_3} I_1$$

By KVL,

$$V_1 = I_1 R_1 + V_x$$

$$= I_1R_1 + (I-\alpha)\left(\frac{R_2R_3}{R_2+R_3}\right)I_1$$

$$\therefore \qquad h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \left[R_1 + \frac{(1-\alpha)R_2R_3}{R_2+R_3}\right]$$

By KCL at the node y,

$$\frac{0-V_x}{R_3} = I_2 + \alpha I_1$$

$$\Rightarrow \qquad I_2 = -\alpha I_1 - (1-\alpha)\left(\frac{R_2R_3}{R_2+R_3}\right)I_1$$

$$= -I_1\left(\frac{R_2+\alpha R_3}{R_2+R_3}\right)$$

$$\therefore \qquad h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = -\left(\frac{R_2+\alpha R_3}{R_2+R_3}\right)$$

Alternate Solution:

Hybrid parameters of a network is given as

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}\begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

$$V_1 = (R_1 + R_3)I_1 + R_2I_2 \qquad \text{...(i)}$$

$$V_2 = (R_2 + \alpha R_3)I_1 + (R_2 + R_3)I_2 \qquad \text{...(ii)}$$

From equation (ii),

$$I_2 = \frac{-(R_2+\alpha R_3)}{R_2+R_3} + \frac{V_2}{R_2+R_3} \qquad \text{...(iii)}$$

Putting equation (iii) in equation (ii),

$$V_1 = (R_1+R_2)I_1 - \frac{R_2(R_2+\alpha R_3)}{R_2+R_3}I_1 + \frac{R_2}{R_2+R_3}V_2$$

$$V_1 = \left[\frac{(R_1+R_2)(R_2+R_3) - R_2(R_2+\alpha R_3)}{R_2+R_3}\right]I_1 + \frac{R_2}{R_2+R_3}V_2 \qquad \text{...(iv)}$$

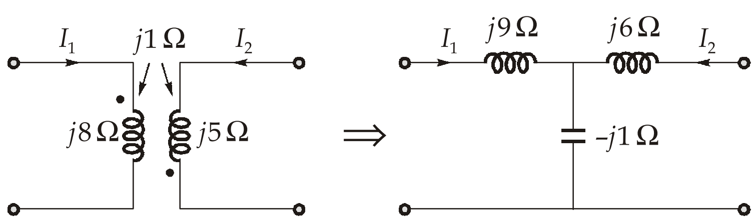
$$I_2 = \frac{-(R_2 + \alpha R_3)}{R_2 + R_3} I_1 + \frac{V_2}{R_2 + R_3} \quad \text{...(v)}$$

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{(R_1 + R_2)(R_2 + R_3) - R_2(R_2 + \alpha R_3)}{R_2 + R_3} & \dfrac{R_2}{R_2 + R_3} \\ \dfrac{-(R_2 + \alpha R_3)}{R_2 + R_2} & \dfrac{1}{R_2 + R_3} \end{bmatrix}$$

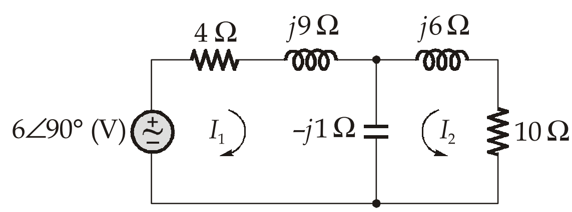
Q.2 (b) (i) Solution:

Using the T-equivalent circuit for the linear transformer, the inductances are

$$X_a = X_1 + X_m = j9\ \Omega$$
$$X_b = X_2 + X_m = j6\ \Omega$$
$$X_c = X_m = -j1\ \Omega$$



Now inserting the T-equivalent circuit in the original circuit, the modified circuit involving no mutual coupling is shown in figure below,



By KVL,

$$j6 = I_1(4 + j9 - j1) + I_2(-j1)$$

$$\Rightarrow \quad (4 + j8)I_1 + (-j1)I_2 = j6$$

$$0 = I_1(-j1) + I_2(10 + j6 - j1)$$

$$\Rightarrow \quad (-j1)I_1 + (10 + j5)I_2 = 0$$

Solving for I_1 and I_2, we get

$$I_1 = \frac{\begin{vmatrix} j6 & -j1 \\ 0 & (10 + j5) \end{vmatrix}}{\begin{vmatrix} (4 + j8) & -j1 \\ -j1 & (10 + j5) \end{vmatrix}} = \frac{-30 + j60}{j100 + 1} = 30\left(\frac{-1 + j2}{j100 + 1}\right)$$

$$= 0.67\angle 27.14° \text{ (A)}$$

$$I_2 = \frac{\begin{vmatrix} (4+j8) & j6 \\ -j1 & 0 \end{vmatrix}}{\begin{vmatrix} (4+j8) & -j1 \\ -j1 & (10+j5) \end{vmatrix}} = \frac{-6}{j100+1} = 0.06\angle -90° \text{ (A)}$$

Q.2 (b) (ii) Solution:

Here,
$$\omega = 300 \text{ rad/sec},$$
$$R = 10\ \Omega,$$
$$C = 30 \times 10^{-6} \text{ F}$$

$\therefore$ Resonance frequency, $\omega_c = 3 \times \omega = 900$ rad/s

1. $\because$
$$\frac{1}{\sqrt{LC}} = 900$$

$\Rightarrow$
$$L = \frac{1}{900^2 \times 30 \times 10^{-6}} = 41.152 \text{ mH}$$

2. For 1st Harmonic, ($\omega = 300$ rad/s)

$$R = 10, \quad X_{L1} = j300 \times 41.152 \times 10^{-3} = j12.3456\ \Omega$$

$$X_{C1} = \frac{-j}{300 \times 30 \times 10^{-6}} = -j111.11\ \Omega$$

$\therefore$
$$Z_1 = 10 + j(12.3456 - 111.11)$$
$$= (10 - j98.77)\Omega = 99.27\angle 95.78°\ \Omega$$

$\therefore$
$$I_1 = \frac{2000}{Z_1} = \frac{2000}{99.27\angle 95.78°} = 20.15\angle -95.78° \text{ (A)}$$

For 3rd Harmonic, ($\omega = 900$ rad/s)

$$R = 10\ \Omega,$$
$$X_{L3} = j900 \times 41.152 \times 10^{-3} = j37.037\ \Omega,$$
$$X_{C3} = \frac{-j}{900 \times 300 \times 10^{-6}} = -j37.037\ \Omega = X_L$$

$\therefore$
$$Z_3 = 10 + j(37.037 - 37.037) = 10\ \Omega$$

$\therefore$
$$I_3 = \frac{400}{Z_3} = \frac{400}{10\angle 0°} = 40\angle 0° \text{ (A)}$$

For 5th Harmonic, ($\omega = 1500$ rad/sec)

$$R = 10\ \Omega,$$

$$X_{L5} = j1500 \times 41.152 \times 10^{-3} = j61.728\ \Omega,$$

$$X_{C5} = \frac{-j}{1500 \times 30 \times 10^{-6}} = -j22.22\ \Omega$$

$$\therefore \quad Z_5 = 10 + j(61.728 - 22.22)$$

$$= (10 - j39.508)\Omega = 40.752\angle 75.795°\ \Omega$$

$$\therefore \quad I_5 = \frac{100}{Z_5} = \frac{100}{40.752\angle 75.795°} = 2.454\angle -75.795°\ \text{(A)}$$

$\therefore$ rms value of the current,

$$I_{rms} = \sqrt{\frac{I_1^2 + I_3^2 + I_5^2}{2}} = \sqrt{\frac{(20.15)^2 + 40^2 + (2.454)^2}{2}}$$

$$= 31.72\ \text{(A)}$$

$\therefore$ rms value of the voltage,

$$V_{rms} = \sqrt{\frac{V_1^2 + V_3^2 + V_5^2}{2}} = \sqrt{\frac{(2000)^2 + (400)^2 + (100)^2}{2}}$$

$$= 1443.95\ \text{(V)}$$

Q.2 (c) Solution:

(i) Applying KVL in the left most loop,

$$V_0 - 100\, I_0 - 20(I_0 - I_1) = 0$$

$$V_0 = 100 I_0 + 20(I_0 - I_1)$$

where, $\quad I_1 = -10 I_0$

$$V_0 = 100 I_0 + 20(I_0 + 10 I_0)$$

$$V_0 = 100\, I_0 + 220\, I_0$$

$$I_0 = \frac{V_0}{320}\text{A}$$

Also, $\quad I_1 = -10 I_0 = \dfrac{-10V_0}{320} = \dfrac{-V_0}{32}$

From the theory of ideal transformer,

$$I_2 = \frac{-I_1}{n} = \frac{-I_1}{\left(\frac{1}{10}\right)} = +\frac{V_0}{32}(10) = \frac{10V_0}{32}$$

Thus, $\quad V_2 = -I_2 R_L = -10 \times \left(\dfrac{10V_0}{32}\right) = \dfrac{-100}{32} V_0$

or Voltage gain $= \frac{V_2}{V_0} = \frac{-100}{32} = -3.125$

To find power gain, first let us calculate the power supplied (instantaneous) by the independent voltage source (V_0)

$$\therefore \quad P_0 = V_0 I_0 = V_0 \times \frac{V_0}{320} = \frac{V_0^2}{320}$$

Power absorbed by the load resistor (R_L) is

$$P_2 = \frac{V_2^2}{10} = \frac{\left(-\frac{100}{32}V_0\right)^2}{10} = \frac{10^3 V_0^2}{1024}$$

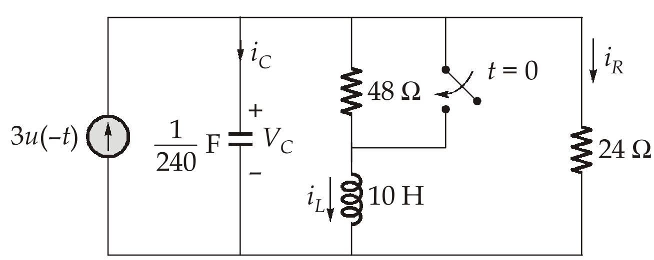
$$\therefore \quad \text{Power gain} = \frac{P_2}{P_0} = \frac{10^3}{1024} \times 320 = 312.5$$

So, for the given problem

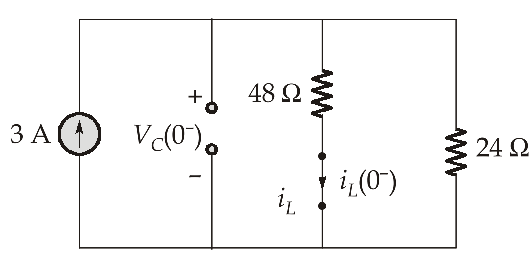
Voltage gain = –3.125

and Power gain = 312.5

(ii) Given circuit,



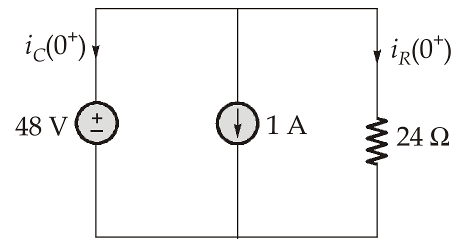
Before $t = 0$, a current source of 3 A is connected and the switch is open. At steady state i.e. at $t = 0^-$, capacitor acts as open circuit and inductor acts as short circuit. The circuit in the steady state for $t < 0$ circuit can be drawn as below:



1. $i_L(0^-) = \frac{3 \times 24}{48 + 24} = 1\text{ A} = i_L(0^+)$

2. $V_C(0^-) = 48 \times i_L(0^-) = 48 \times 1 = 48\text{ V} = V_C(0^+)$

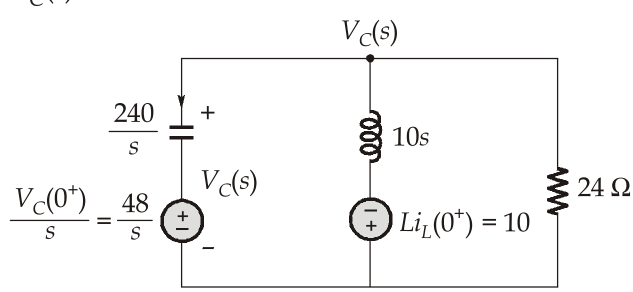
At $t > 0$, the switch is closed and the current source is deactivated (open). The circuit at $t = 0^+$ can be drawn as



3. $i_R(0^+) = \frac{48}{24} = 2\text{ A}$

4. Using KCL, $i_C(0^+) = -1 - 2 = -3\text{ A}$

5. To calculate $V_C(t)$, the circuit in s-domain for $t > 0$ is drawn as below:



by nodal analysis,

$$\frac{V_C(s) - \frac{48}{s}}{\frac{240}{s}} + \frac{V_C(s) + 10}{10s} + \frac{V_C(s)}{24} = 0$$

$$V_C(s)\left[\frac{s}{240} + \frac{1}{10s} + \frac{1}{24}\right] = \frac{1}{5} - \frac{1}{s}$$

$$V_C(s)\left[\frac{s^2 + 24 + 10s}{240s}\right] = \frac{s-5}{5s}$$

$$V_C(s) = \frac{48s - 240}{s^2 + 10s + 24} = \frac{48(s-5)}{(s+4)(s+6)}$$

By using partial fraction expansion

$$\therefore \qquad \frac{48(s-5)}{(s+4)(s+6)} = \frac{A}{s+4} + \frac{B}{s+6}$$

$$A = \left.\frac{48(s-5)}{s+6}\right|_{s=-4} = \frac{48(-4-5)}{2} = -216$$

$$B = \left.\frac{48(s-5)}{s+4}\right|_{s=-6} = \frac{48(-6-5)}{-2} = 264$$

$$\therefore \quad V_C(t) = -216e^{-4t} + 264e^{-6t}\,; \quad t > 0$$

at $t = 0.2$,

$$V_C(0.2) = -216e^{-0.8} + 264\,e^{-1.2}$$

$$\therefore \quad V_C(0.2) = -17.54 \text{ V}$$

Q.3 (a) (i) Solution:

Data:

$$V_L = 415\ \Omega, \qquad f = 50 \text{ Hz}$$
$$R = 15\ \Omega, \qquad C = 177\ \mu\text{F}$$
$$L = 0.1 \text{ H},$$

For a star-connected load,

$$V_{\text{ph}} = \frac{V_L}{\sqrt{3}} = \frac{415}{\sqrt{3}} = 239.6 \text{ V}$$

$$X_L = 2\pi fL = 2\pi \times 50 \times 0.1 = 31.42\ \Omega$$

$$X_C = \frac{1}{2\pi fL} = \frac{1}{2\pi\times 50\times 177\times 10^{-6}} = 17.98\ \Omega$$

$$\bar{Z}_{ph} = R + jX_L - jX_C = 15 + j31.42 - j17.98$$
$$= 15 + j13.44 = 20.14\angle 41.86°\ \Omega$$

$$Z_{\text{ph}} = 20.14\ \Omega$$
$$\phi = 41.86°$$

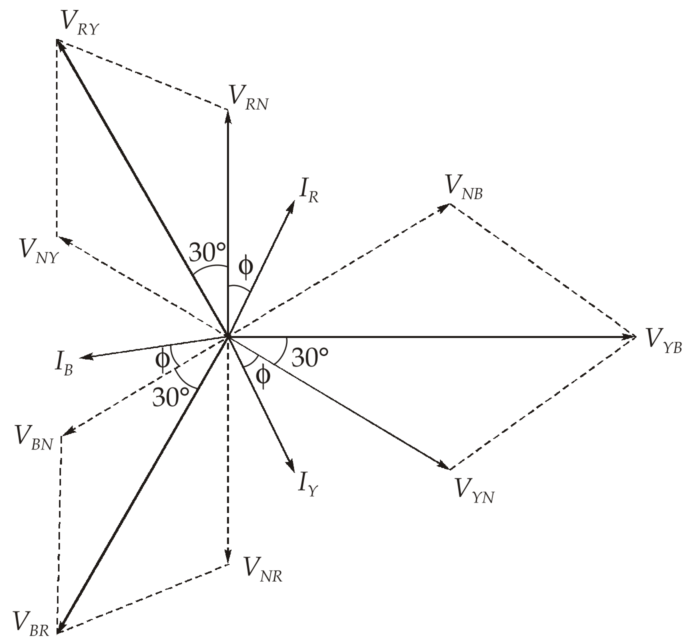
$$\text{Power factor} = \cos\phi = \cos(41.86°) = 0.744 \text{ (lagging)}$$

$$I_{\text{ph}} = \frac{V_{\text{ph}}}{Z_{\text{ph}}} = \frac{239.6}{20.14} = 11.9 \text{ A}$$

$$P = \sqrt{3}\,V_L I_L \cos\phi$$
$$= \sqrt{3}\times 415\times 11.9\times 0.744 = 6.36 \text{ kW}$$

$$Q = \sqrt{3}\times 415\times 11.9\times \sin(41.86°) = 5.71 \text{ kVAR}$$

$$S = \sqrt{3}V_L I_L$$
$$= \sqrt{3}\times 415\times 11.9 = 8.55 \text{ kVA}$$



$$\phi = 41.86°$$

If the same impedances are connected in delta,

$$V_L = V_{ph} = 415 \text{ V}$$

$$Z_{ph} = 20.14\ \Omega$$

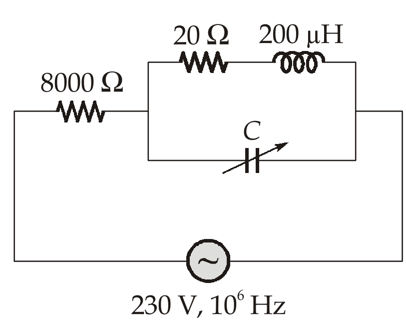
$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{415}{20.14} = 20.61 \text{ A}$$

$$I_L = \sqrt{3} I_{ph} = \sqrt{3} \times 20.61 = 35.69 \text{ A}$$

$$P = \sqrt{3}\, V_L I_L \cos\phi$$

$$= \sqrt{3} \times 415 \times 35.69 \times 0.744 = 19.09 \text{ kW}$$

Q.3 (a) (ii) Solution:



Data: $R = 20\ \Omega$, $L = 200\ \mu\text{H}$

$$f = 10^6 \text{ Hz}, \qquad V = 230 \text{ V}$$

$$R_S = 8000\ \Omega$$

$$X_L = 2\pi f L = 2 \times \pi \times 10^6 \times 200 \times 10^{-6} = 1256.6\ \Omega$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$10^6 = \frac{1}{2\pi}\sqrt{\frac{1}{200\times10^{-6}\times C} - \frac{(20)^2}{(200\times10^{-6})^2}}$$

$$C = 126.65 \times 10^{-12} \text{ F} = 126.65 \text{ pF}$$

Quality factor, $$Q_0 = \frac{2\pi f L}{R} = \frac{2\pi\times10^6\times200\times10^{-6}}{20} = 62.83$$

Dynamic impedance, $$Z = \frac{L}{CR} = \frac{200\times10^{-6}}{126.65\times10^{-12}\times20} = 78958\ \Omega$$

Total equivalent impedance of the circuit at resonance

$$= 78958 + 8000 = 86958\ \Omega$$

Total circuit current $$= \frac{230}{86958} = 2.645 \times 10^{-3} \text{ A} = 2.65 \text{ A}$$

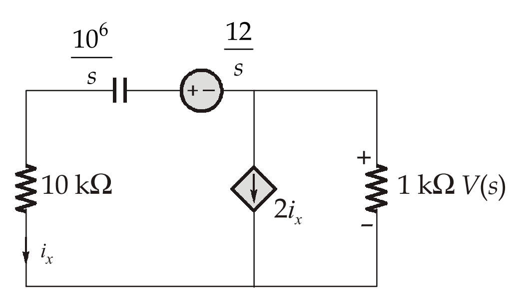
Q.3 (b) (i) Solution:

For $t < 0$,

Initially switch is closed, $$V_c(0^-) = 10000\, i_x + 1000\,(2i_x) = 12000 i_x$$

$$= 12000 \times \frac{10}{10000} = 12 \text{ V}$$

Converting circuit in laplace domain,



$$3i_x = -V(t)$$

$$10ki_x + 3ki_x + \frac{10^6}{s} i_x = \frac{12}{s}$$

$$s(13 \times 10^3\ i_x) + 10^6\ i_x = 12$$

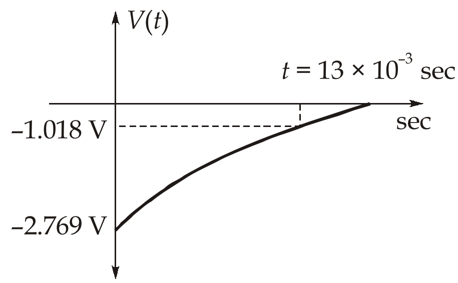
$$i_x = \frac{12}{13\times10^3 s + 10^6} = \frac{\frac{12}{13}\times10^{-3}}{s+\frac{10^3}{13}}$$

$$i_x(t) = 0.923 \times 10^{-3}\ e^{-76.92t}$$

$$V(t) = -3i_x(t)$$

$$= -2.769\ e^{-76.92t}\ \text{V}$$

At $t = 0$, $\quad V(0) = -2.769$ volts



Q.3 (b) (ii) Solution:

1. Since no voltage is applied to either inductor for $t < 0$, therefore, $i_1(0) = 0$, no voltage appears across the second inductor until $t \geq 1$. Hence, $i_2(1) = 0$. Now, for $0 \leq t < 1$,

$$i_s(t) = i_1(t) = \frac{1}{L_1}\int_0^t v_s(\tau)d\tau$$

$$= \int_0^t \cos(\tau)d\tau = \sin(t)A$$

At $t = 1$, the switch closes. The two inductors are then in parallel, and the source voltage appears across each. Hence, using equation

$$i_L(t) = i_1(t_0) + \frac{1}{L}\int_{t_0}^t v_L(\tau)d\tau$$

We have, $$i_1(t) = i_1(1) + \int_1^t \cos(\tau)d\tau = \sin(1) + \sin(t) - \sin(1) = \sin(t)A$$

Using the same equation, we get

$$i_2(t) = i_2(1) + \int_1^t \cos(\tau)d\tau = \sin(t) - \sin(1)A$$

From KCL, the input current

$$i_s(t) = i_1(t) + i_2(t) = 2\sin(t) - \sin(1)A \text{ for } t \geq 1$$

2. Using equation, for the stored energy during the interval $[t_0, t_1]$,

$$W_L[t_0, t_1] = \frac{1}{2}L\left[i_L^2(t_1) - i_L^2(t_0)\right]\text{Joule,}$$

For $0 \le t \le 1$, $\quad W_{L1}(0, t) = 0.5 \sin^2 t$ J

$$W_{L2}(0, t) = 0$$

and, for $t \ge 1$, $\quad W_{L1}(0, t) = 0.5 \sin^2 t$ J

$$W_{L2}(0, t) = 0.5[\sin^2(t) - 2\sin(1)\sin(t) + \sin^2(1)] \text{ Joule}$$

Q.3 (c) Solution:

For $t < 0$, switches S_1 and S_2 are open.

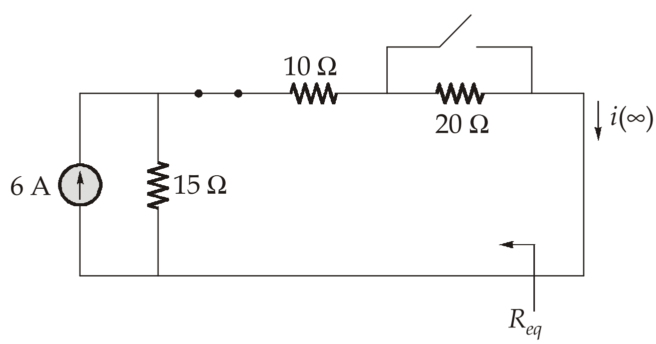
So that $i(0) = 0$ A

Since the inductor current cannot change instantly,

$$i(0^-) = i(0) = i(0^+) = 0\text{A}$$

For $0 \le t \le 2$, S_1 is closed,

S_2 is opened; 10 Ω and 20 Ω resistors are in series. Hence, assuming for now that S_1 is closed forever.



$$\therefore \quad i(\infty) = \frac{6 \times 15}{30 + 15} = 2 \text{ A}$$

$$R_{eq} = 15 + 30 = 45\ \Omega$$

$$\text{time constant, } \tau = \frac{L}{R_{eq}} = \frac{5}{45} = \frac{1}{9}\text{sec}$$

Thus, $\quad i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} = 2 + [0 - 2]e^{-9t}$

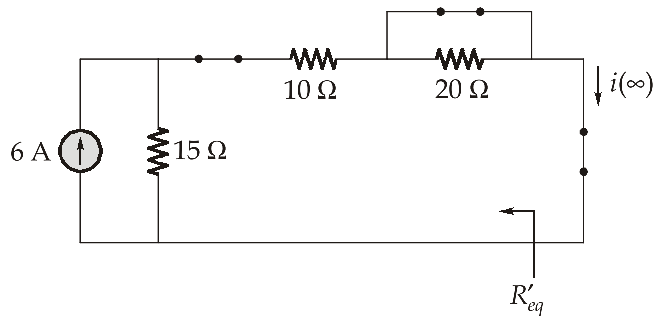
$$\therefore \quad i(t) = 2 - 2e^{-9t} \text{ A, } 0 \le t \le 2$$

For $t \ge 2$, switch S_2 is closed, 20 Ω resistor is shorted, but sudden change does not affect the inductor current because current cannot change abruptly.

Thus, the initial current,

$$i(2) = i(2^-) = 2(1 - e^{-18}) \simeq 2 \text{ A}$$

$i(\infty)$ after S_2 closed is calculated as



$$\therefore \qquad i(\infty) = \frac{6\times 15}{10+15} = 3.6 \text{ A}$$

$$R'_{eq} = 10 + 15 = 25\ \Omega$$

$$\text{time constant, } \tau = \frac{L}{R'_{eq}} = \frac{5}{25} = \frac{1}{5}\text{sec}$$

Hence, $\quad i(t) = i(\infty) + [i(2) - i(\infty)]e^{-(t-2)/\tau};\ t \geq 2$

We need $(t - 2)$ in the exponential because of the time delay.

Thus, $\quad i(t) = 3.6+[2-3.6]e^{-5(t-2)}A$

$$i(t) = 3.6 - 1.6e^{-5(t-2)} \text{ A};\ t > 2$$

$\therefore \quad i(t)$ for all t is

$$i(t) = \begin{cases} 0 & ;\quad t<0 \\ 2(1-e^{-9t}) & ;\ 0<t<2 \\ 3.6-1.6e^{-5(t-2)} & ;\quad t>2 \end{cases}$$

at $t = 1$ sec, $\quad i(1) = i(t)\,|_{t=1\text{ sec},}\ 0 < t < 2 = 2(1 - e^{-9}) = 1.9997$ A

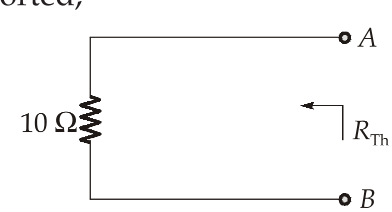
at $t = 3$ sec, $\quad i(3) = i(t)\,|_{t=3\text{ sec},}\ t > 2 = 3.6 - 1.6e^{-5(3-2)} = 3.589$ A

Q.4 (a) Solution:

To calculate Thevenin equivalent, R_{Th} and V_{Th} across AB, we remove 5 Ω resistor.

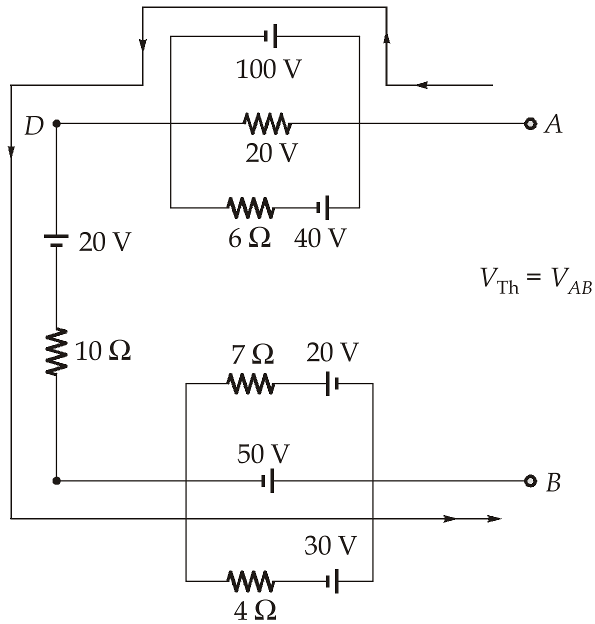
Calculation of R_{Th}:

All voltage source will be shorted,



Hence, $\quad R_{Th} = 10\ \Omega$

Calculation of V_{Th}:

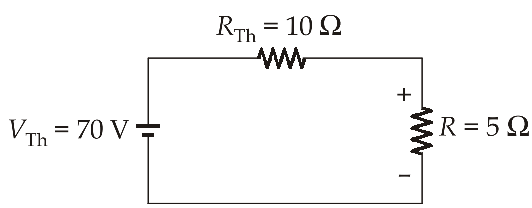


Applying KVL in the path shown,

$$V_A - 100 - 20 + 50 = V_B$$

$$V_{Th} = V_A - V_B = 70\text{ V}$$

Thevenin's equivalent circuit:

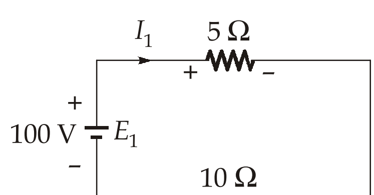


The current through 5 Ω resistor is given by,

$$I_R = \frac{70}{15} = 4.667\text{ A}$$

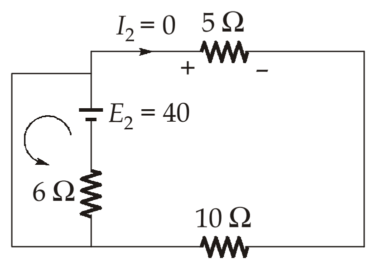
Verification by superposition theorem: According to superposition theorem, in any linear and bilateral network having multiple independent sources, the response of an element will be equal to the algebraic sum of the responses of that element by considering one source at a time and deactivating all other sources.

- **Current due to E_1,**



$$I_1 = \frac{100}{15} = 6.67 \text{ A}$$

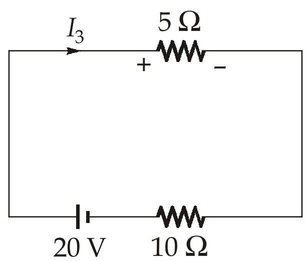
- **Current due to E_2,**



Current will flow through shorted branch. Hence,

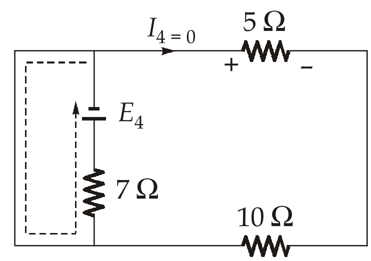
$$I_2 = 0 \text{ A}$$

- **Current due to E_3,**



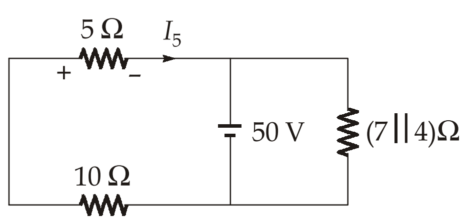
$$I_3 = \frac{20}{15} = 1.33 \text{ A}$$

- **Current due to E_4,**



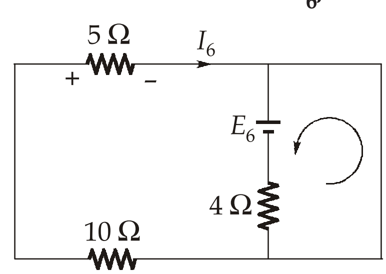
$$I_4 = 0 \text{ A}$$

- **Current due to E_5,**



$$I_5 = \frac{-50}{15} = -3.33 \text{ A}$$

- **Current due to E_6,**



$$I_6 = 0 \text{ A}$$

Note: Resistance in parallel to the ideal voltage source has no effect on current I.

Applying superposition theorem:

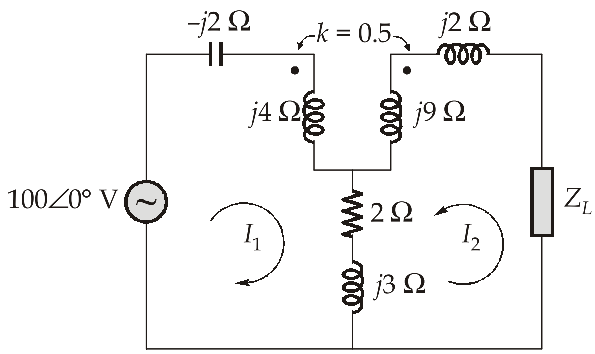
Current through 5 Ω resistor,

$$\begin{aligned} I &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \\ &= 6.67 + 0 + 1.33 + 0 - 3.33 + 0 = 4.67 \text{ A} \end{aligned}$$

Thus, same results are obtained using both Thevenin and superposition theorem.

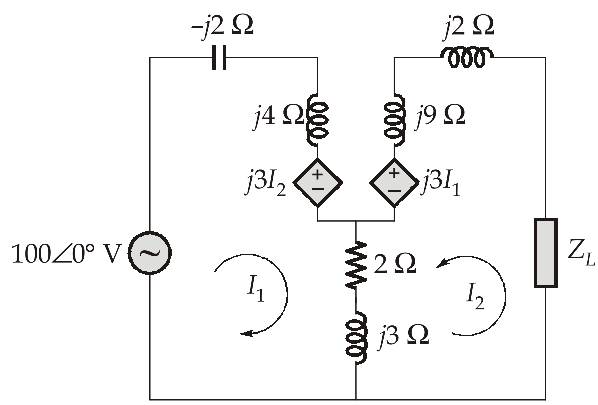
Q.4 (b) Solution:

(i)

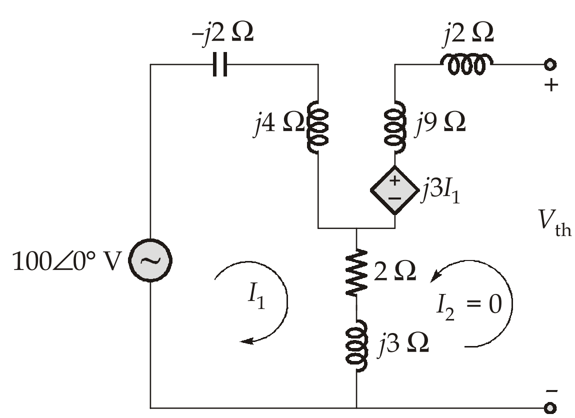


$$\text{Mutual inductance} = k\sqrt{L_1L_2} = 0.5\sqrt{4\times 9} = 3\text{ H}$$

Using the dot convention, the equivalent circuit can be drawn as below:



Thevenin voltage: To calculate Thevenin voltage V_{th}, we disconnect the load impedance Z_L.



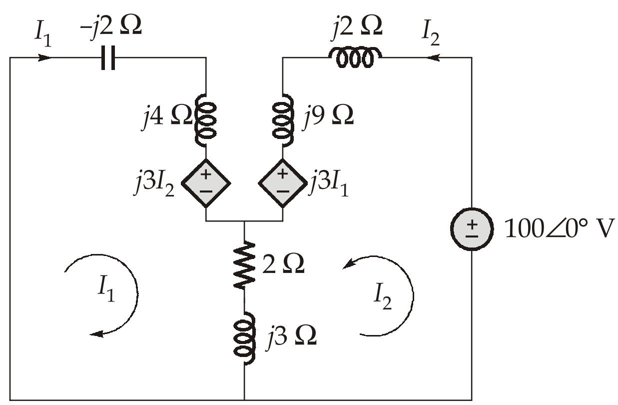
Applying KVL in loop 1, we get,

$$100 = (-j2 + 4j + 2 + j3)I_1$$

$$I_1 = \frac{100}{j2+5} = 18.57\angle -68.19°\text{A}$$

$$V_{th} = j3I_1 + (2 + j3)I_1$$
$$V_{th} = (2 + j6)I_1$$
$$V_{th} = (2 + j6) \times 18.57 \angle{-68.19°}$$
$$V_{th} = 117.44 \angle 3.36° \text{V}$$

Thevenin impedance: To calculate Thevenin impedance Z_{th}, we replace the independent voltage source by short-circuit. Applying 100 V at the terminals of Z_L, the Thevenin impedance is given by,



Thevenin impedance: $$Z_{th} = \frac{100}{I_2}$$

Applying KVL in loop 1, we get

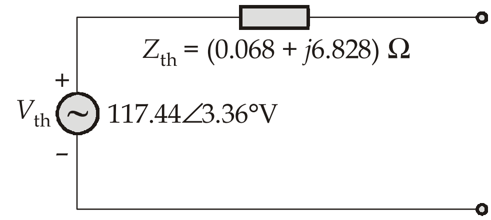
$$(-j2 + j4)I_1 + j3I_2 + (2 + j3)(I_1 + I_2) = 0$$
$$(2 + j5)I_1 = -(2 + j6)I_2$$
$$I_1 = -\frac{(2+j6)}{(2+j5)}I_2$$

Applying KVL in loop 2, we get

$$(2 + j3)(I_1 + I_2) + j3I_1 + j11I_2 = 100$$
$$(2 + j6)I_1 + (2 + j14)I_2 = 100$$
$$-(2+j6)\left(\frac{2+j6}{2+j5}\right)I_2 + (2+j14)I_2 = 100$$
$$I_2 = \frac{-100}{\frac{(2+j6)^2}{2+j5} - (2+j14)}$$
$$Z_{th} = \frac{100}{I_2} = (2+j14) - \frac{-32+24j}{2+j5}$$

$$Z_{th} = (0.068 + j6.828)\ \Omega$$

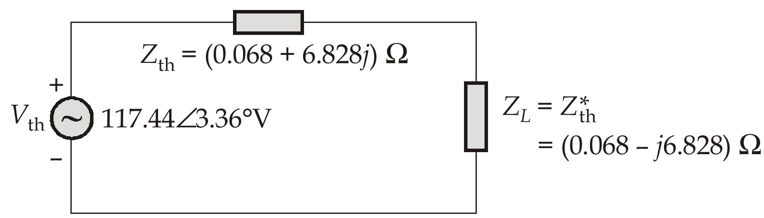
Thevenin equivalent circuit is



(ii) For maximum power transfer, according to maximum power transfer theorem,

$$Z_L = Z_{th}^*$$

$$Z_L = (0.068 - j6.828)\Omega$$

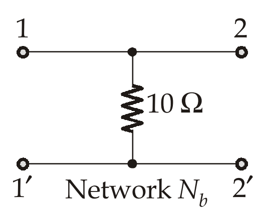


The maximum power transferred to the load Z_L is thus,

$$P_{max} = \frac{|V_{th}|^2}{4R_{th}} = \frac{117.44 \times 117.44}{4 \times 0.068} = 50706.45\text{ W} = 50.7\text{ kW}$$

Q.4 (c) (i) Solution:

The given two, two-port networks are in series with the source. For the network N_b,



Network N_b

$$z_{12b} = z_{21b} = 10\ \Omega = z_{11b} = z_{22b}$$

Thus,

$$[z] = [z_a] + [z_b] = \begin{bmatrix} 12 & 8 \\ 8 & 20 \end{bmatrix} + \begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix} = \begin{bmatrix} 22\ \Omega & 18\ \Omega \\ 18\ \Omega & 30\ \Omega \end{bmatrix}$$

But

$$V_1 = z_{11}I_1 + z_{12}I_2 = 22I_1 + 18I_2 \quad \text{...(i)}$$

$$V_2 = z_{21}I_1 + z_{22}I_2 = 18I_1 + 30I_2 \quad \text{...(ii)}$$

Also, at the input port

$$V_1 = V_s - 5I_1 \quad \text{...(iii)}$$

and at the output port

$$V_2 = -20I_2$$

$$\Rightarrow \qquad I_2 = -\frac{V_2}{20} \qquad \text{...(iv)}$$

Substituting equation (iii) and (iv) into equation (i) gives

$$V_s - 5I_1 = 22I_1 - \frac{18}{20}V_2$$

$$\Rightarrow \qquad V_s = 27I_1 - 0.9\,V_2 \qquad \text{...(v)}$$

While substituting equation (iv) into equation (ii) yields

$$V_2 = 18I_1 - \frac{30}{20}V_2$$

$$\Rightarrow \qquad I_1 = \frac{2.5}{18}V_2 \qquad \text{...(vi)}$$

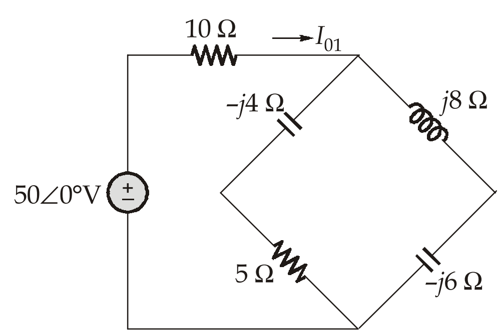
Substituting equation (vi) into equation (v), we get

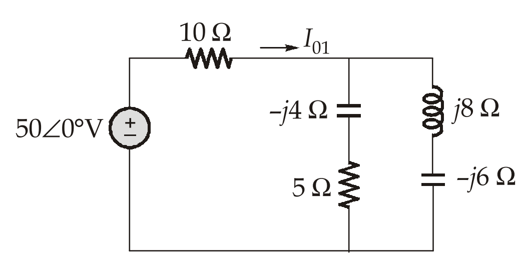
$$V_s = 27 \times \frac{2.5}{18}V_2 - 0.9\,V_2 = 2.85\,V_2$$

$$\frac{V_2}{V_s} = \frac{1}{2.85} = 0.351$$

Q.4 (c) (ii) Solution:

By using superposition theorem, first considering voltage source and open circuiting current source. Let current due to voltage source is I_{01}.





Let,
$$Z_{eq} = (5 - j4) \parallel (j8 - j6)$$

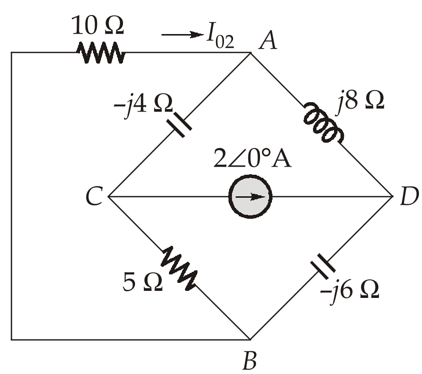
$$= \frac{(5 - j4) \times j2}{5 - j4 + j2}$$

$$Z_{eq} = \frac{8+j10}{5-j2}$$

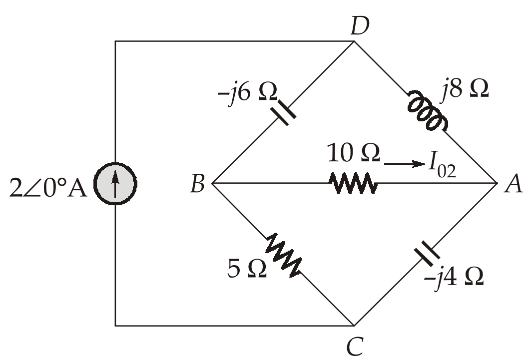
$$\text{Current } I_{01} = \frac{50\angle 0^\circ}{10+Z_{eq}} = \frac{50\angle 0^\circ}{10+\dfrac{8+j10}{5-j2}}$$

$$\therefore \quad I_{01} = 4.575\angle -12.019^\circ \text{ A}$$

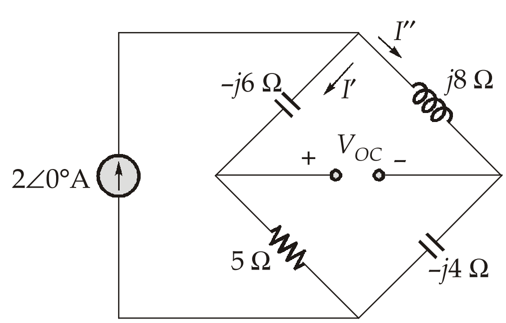
Now, considering current source and short circuiting voltage source. Let the current due to the current source be I_{02}.



By redrawing the above bridge circuit, we get



By using Thevenin's theorem,



$$V_{OC} = 5I' - (-j4)I''$$

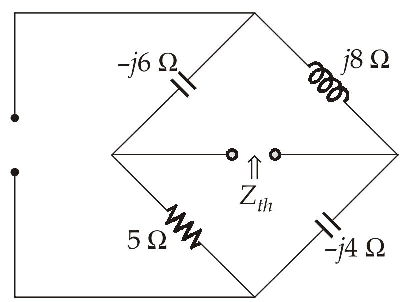
where, $$I' = \frac{2\angle 0° \times (j4)}{5 - j6 + j4} = 1.485\angle 111.80°\text{A}$$

$$I'' = \frac{2\angle 0° \times (5 - j6)}{5 - j6 + j4} = 2.90\angle -28.4°\text{A}$$

$\therefore$ $$V_{OC} = 7.425 \angle 111.80° - 11.60 \angle -118.4°\text{A}$$

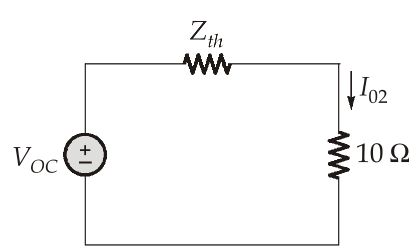
$$V_{OC} = 17.32 \angle 80.83° \text{ V}$$

Thevenin's resistance, Z_{th}:



$\therefore$ $$Z_{\text{th}} = (j2) \parallel (5 - j4)$$

$$Z_{\text{th}} = 2.378 \angle 73.141° \ \Omega$$



$\therefore$ $$I_{02} = \frac{V_{OC}}{Z_{th} + 10} = \frac{17.32\angle 80.83°}{(2.378\angle 73.141°) + 10}$$

$$I_{02} = 1.584 \angle 68.81°\text{A}$$

According to superposition theorem,

$$I_0 = I_{01} + I_{02}$$

$$= 4.575 \angle -12.019° + 1.584 \angle 68.81°\text{A}$$

$$I_0 = 5.075 \angle 5.93°\text{A}$$

Section B : Systems & Signal Processing

Q.5 (a) Solution:

The Fourier transform $X(e^{j\omega})$ of the sequence $x(n)$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

A change of variables is performed by letting $m = (n + N_1)$, which also yields $n = (m - N_1)$, $m = 0$ as $n = -N_1$ and $m = 2N_1$ as $n = N_1$. Therefore,

$$X(e^{j\omega}) = \sum_{m=0}^{2N_1} e^{-j\omega(m-N_1)}$$

$$X(e^{j\omega}) = e^{j\omega N_1}\sum_{m=0}^{2N_1} e^{-j\omega m}$$

$$= e^{j\omega N_1}\left(\frac{1-e^{-j\omega(2N_1+1)}}{1-e^{-j\omega}}\right)$$

$$= e^{j\omega N_1}\left(\frac{e^{-j\omega/2}(e^{j\omega/2}-e^{-j\omega(2N_1+1/2)})}{e^{-j\omega/2}(e^{j\omega/2}-e^{-j\omega/2})}\right)$$

$$= \frac{(e^{j\omega(N_1+1/2)}-e^{-j\omega(N_1+1/2)})}{(e^{j\omega/2}-e^{-j\omega/2})}$$

$$X(e^{j\omega}) = \frac{\sin(\omega(N_1+1/2))}{\sin(\omega/2)}$$

$$x(n) = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases} \longleftrightarrow X(e^{j\omega}) = \frac{\omega\frac{(N_1+1)}{2}}{\sin(\omega/2)}$$

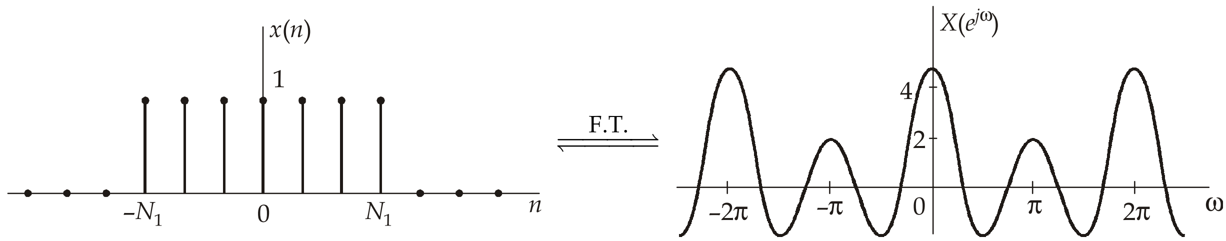
The Fourier transform is sketched in figure (b) below for $N_1 = 2$. Since $X(e^{j\omega})$ is real, the magnitude and phase spectrum are given by

$$|X(e^{j\omega})| = \left|\frac{\sin(\omega(N_1+1/2))}{\sin(\omega/2)}\right|$$

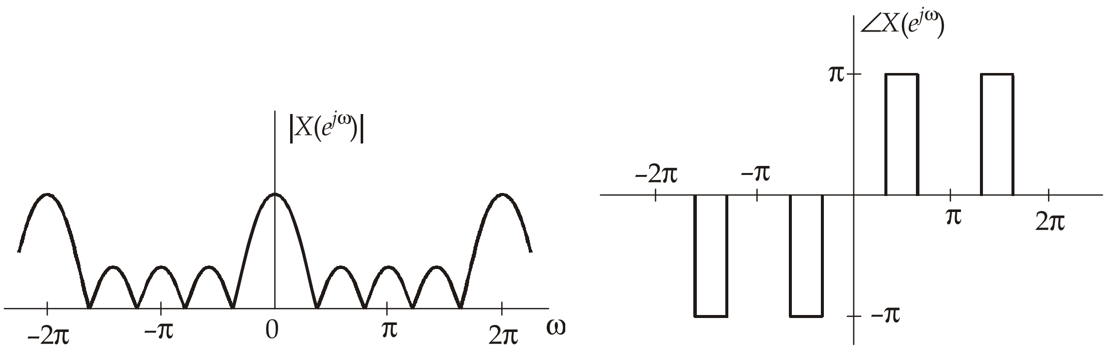
and

$$\angle X(e^{j\omega}) = \begin{cases} 0, & X(e^{j\omega}) > 0 \\ \pi, & X(e^{j\omega}) < 0 \end{cases}$$

The magnitude and phase spectrum are plotted, respectively, in figure (c) and (d) for $N_1 = 2$.



(a) **(b)**



(c) **(d)**

Q.5 (b) Solution:

(i) Given that, $x(t) = \cos(100\pi t)$

Comparing with $x(t) = \cos(\omega_1 t)$

We have, $\omega_1 = 100\pi$ or $f_1 = 50$ Hz

Therefore, the Nyquist rate is given by

$$\omega_s = 2\omega_1 = 200\pi \quad \text{or} \quad f_s = 2f_1 = 100 \text{ Hz}$$

(ii) If the signal is sampled at $f_s = 200$ Hz, the discrete-time signal is

$$x(t)\big|_{t=nT_s} = x(nT_s) = x(n) = \cos(100\pi n T_s)$$

$$= \cos\left(\frac{100\pi}{200}n\right)$$

$$x(n) = \cos\left(\frac{\pi}{2}n\right)$$

(iii) If the signal is sampled at $f_s = 75$ Hz, the discrete-time signal is

$$x(t)\big|_{t=nT_s} = x(nT_s) = x(n) = \cos(100\pi n T_s)$$

$$= \cos\left(\frac{100\pi}{75}n\right)$$

$$x(n) = \cos\left(\frac{4\pi}{3}n\right) = \cos\left(2n\pi - \frac{2\pi}{3}n\right)$$

$$x(n) = \cos\left(\frac{2\pi}{3}n\right)$$

The frequency of the discrete-time signal $x(n)$ is $f_d = 1/3$ cycles/sample.

(iv) For the sampling rate $f_s = 75$ Hz, we have

$$f_d = \frac{f}{f_s}$$

or, equivalently, we have $f = f_d f_s$

$$f = \frac{1}{3}75 = 25\text{ Hz}$$

Clearly ,the sinusoidal signal

$$y(t) = \cos(2\pi f t)$$

$$y(t) = \cos(50\pi t)$$

sampled at $f_s = 75$ Hz yields identical samples. Hence, $f = 50$ Hz is an alias of $f = 25$ Hz for the sampling rate $f_s = 75$ Hz.

Q.5 (c) Solution:

By taking the first derivative by $X(z)$, we can convert the given z-transform to a rational expression. Therefore, by taking the first derivative of $X(z)$, we obtain

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1+az^{-1}}$$

Thus,

$$-z\frac{dX(z)}{dz} = \frac{-az^{-1}}{1+az^{-1}}$$

Taking the inverse z-transform yields

$$Z^{-1}\left[-z\frac{dX(z)}{dz}\right] = Z^{-1}\left[\frac{-az^{-1}}{1+az^{-1}}\right]$$

$$nx(n) = Z^{-1}\left[\frac{-az^{-1}}{1+az^{-1}}\right]$$

We know that

$$a^n u(n) \longleftrightarrow \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

Therefore, $$(-a)^n u(n) \longleftrightarrow \frac{1}{1+az^{-1}}, \quad |z|>|a|$$

$$a(-a)^n u(n) \longleftrightarrow \frac{a}{1+az^{-1}}, \quad |z|>|a|$$

Using the time-shifting property, we obtain

$$a(-a)^{n-1} u(n-1) \longleftrightarrow \frac{az^{-1}}{1+az^{-1}}, \quad |z|>|a|$$

$$-(-a)^n u(n-1) \longleftrightarrow \frac{az^{-1}}{1+az^{-1}}, \quad |z|>|a|$$

Consequently, $$nx(n) = -(-a)^n u(n-1)$$

$$x(n) = \frac{-(-a)^n}{n} u(n-1) = \frac{1}{n}(-1)^{n+1} a^n u(n-1)$$

Q.5 (d) Solution:

Conditions for existence of Fourier transform:

(i) 1. Signal $x(t)$ should be absolutely integrable i.e.,

If $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

then $|X(\omega)| < \infty$

2. Signal $x(t)$ should be deterministic over any finite interval.

(a) It should have finite number of maxima and minima within any finite intergal.

(b) The signal must have a finite number of discontinuous within any finite interval, and each discontinuity must be finite.

These conditions are sufficient but not necessary. If a signal satisfies these, it will have a transform, however some signals that violate these conditions can still possess a Fourier transform.

Examples:

- Energy signals like $e^{-a|t|}$ satisfies Dirichlet conditions.
- Signals like a DC constant (A) or a sine wave (i.e., $\sin\omega_0 t$) are not absolutely integrable. However, they possess a Fourier transform in the form of impulse functions.

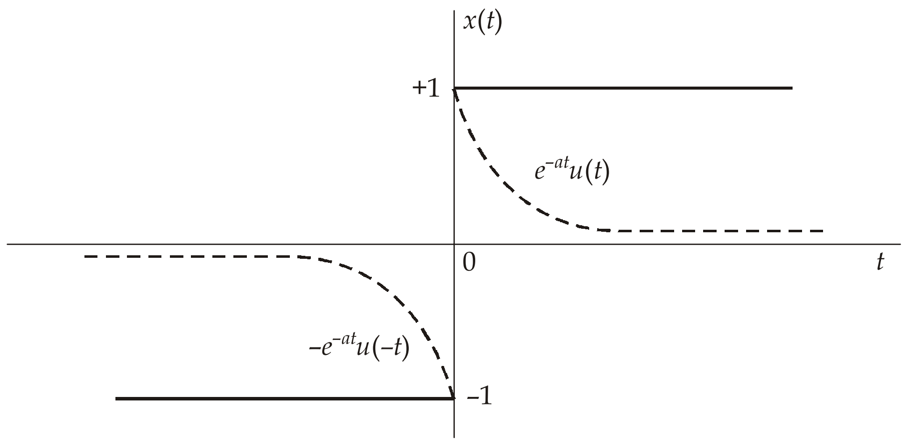
- A signal that is neither an energy nor a power signal, such as $x(t) = tu(t)$, does not have a defined Fourier transform because it grows indefinitely.

Q.5 (e) Solution:

The given signal $x(t) = \text{sgn}(t)$ can be rewritten as

$$\text{sgn}(t) = u(t) - u(-t)$$

This signal is not absolutely integrable. So, we approach this problem by considering sgn(t) to be a sum of exponential $e^{-at}u(t) - e^{at}u(-t)$ in the limit as $a \to 0$.



Thus, $$\text{sgn}(t) = \lim_{a\to 0}\left[e^{-at}u(t) - e^{at}u(-t)\right]$$

and $$X(\omega) = F[\text{sgn}(t)] = \lim_{a\to 0} F\left[e^{-at}u(t) - e^{at}u(-t)\right]$$

$$= \lim_{a\to 0}\left(F\left[e^{-at}u(t)\right] - F\left[e^{-at}u(-t)\right]\right)$$

$$= \lim_{a\to 0}\left(\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right)$$

$$= \lim_{a\to 0}\left(\frac{-2j\omega}{a^2+\omega^2}\right) = \frac{-2j\omega}{\omega^2}$$

$$X(\omega) = F[\text{sgn}(t)] = \frac{2}{j\omega}$$

$$\text{sgn} \longleftrightarrow \frac{2}{j\omega}$$

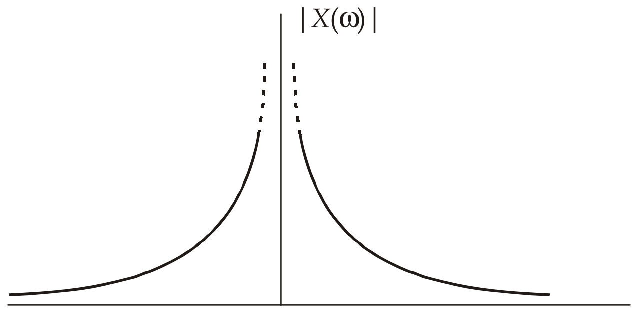
The magnitude spectrum is

$$|X(\omega)| = \left|\frac{2}{\omega}\right|$$

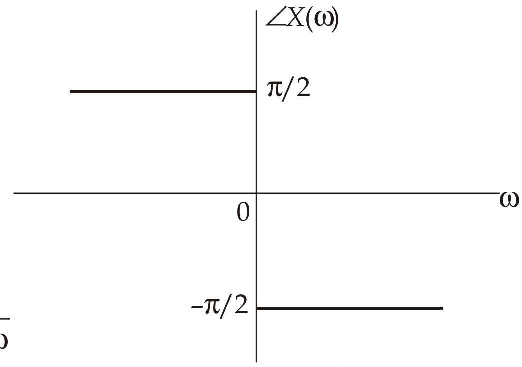
and the phase spectrum is

$$\angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{0}\right) = \begin{cases} -\frac{\pi}{2}, & \omega > 0 \\ \frac{\pi}{2}, & \omega < 0 \end{cases}$$

Figure below shows the magnitude and phase spectrum.



(a)



(b)

Q.6 (a) (i) Solution:

$$y(t) = (2 - 3e^{-t} + e^{-3t})u(t)$$

$$Y(s) = \frac{2}{s} - \frac{3}{s+1} + \frac{1}{s+3}$$

$$Y(s) = \frac{2(s^2+4s+3) - 3s(s+3) + s(s+1)}{s(s+1)(s+3)} = \frac{6}{s(s+1)(s+3)}$$

Consider the given unit step response

$$s(t) = (1 - e^{-t} - te^{-t})u(t)$$

$$s(t) = u(t) - e^{-t}u(t) - te^{-t}u(t)$$

The Laplace transform of this equation yields

$$S(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

We know that, $$H(s) = s\,S(s) = s\left(\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right)$$

$$H(s) = 1 - \frac{s}{s+1} - \frac{s}{(s+1)^2}$$

$$= \frac{(s+1)^2 - s(s+1) - s}{(s+1)^2}$$

$$= \frac{s^2+2s+1-s^2-s-s}{(s+1)^2}$$

$$H(s) = \frac{1}{(s+1)^2} = \frac{Y(s)}{X(s)}$$

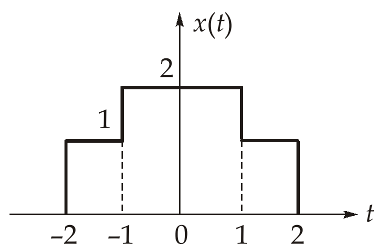
Therefore,
$$X(s) = \frac{Y(s)}{H(s)} = \frac{6/s(s+1)(s+3))}{1/((s+1)^2)} = \frac{6(s+1)}{s(s+3)}$$

$$X(s) = \frac{2}{s} + \frac{4}{s+3}$$

The inverse transform of this equation yields

$$x(t) = 2u(t) + 4e^{-3t}u(t)$$

Q.6 (a) (ii) Solution:



$$x(t) = u(t+2) + u(t+1) - u(t-1) - u(t-2)$$

Fourier transform of the step function is $u(t) \xleftrightarrow{FT} \frac{1}{j\omega} + \pi\delta(\omega)$

Time shifting property is $x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(j\omega)$

So, the Fourier transform of $x(t)$ is

$$X(j\omega) = \left[e^{j2\omega} + e^{j\omega} - e^{-j\omega} - e^{-j2\omega}\right]\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$= \left[\frac{e^{j2\omega}-e^{-j2\omega}}{2j} + \frac{e^{j\omega}-e^{-j\omega}}{2j}\right]\left[\frac{2j}{j\omega} + j2\pi\delta(\omega)\right]$$

$\therefore$
$$X(j\omega) = (\sin 2\omega + \sin\omega)\left(\frac{2}{\omega} + j2\pi\delta(\omega)\right)$$

$$= \frac{2\sin 2\omega}{\omega} + \frac{2\sin\omega}{\omega} = 2[Sa(2\omega) + Sa(\omega)]$$

Q.6 (b) (i) Solution:

The waveform is periodic with period $T = 2$ and fundamental frequency $\omega_0 = 2\pi/T = \pi$.
The given waveform for one period may be written as

$$x(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

Since the wave is neither even nor odd, the series will contain both sine and cosine terms.

$$a_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{2}\left(\int_0^1 t\,dt + \int_1^2 0dt\right) = \frac{1}{4}$$

Now, we determine a_n and b_n:

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)dt$$

$$= \frac{2}{2}\left(\int_0^1 t\cos(n\pi t)dt + \int_1^2 0\cos(n\pi t)dt\right)$$

$$= \int_0^1 t\cos(n\pi t)dt + 0$$

$$= \left[t\frac{\sin(n\pi t)}{n\pi}\right]_0^1 + \left[\frac{\cos(n\pi t)}{(n\pi)^2}\right]_0^1 = \frac{1}{(n\pi)^2}(\cos(n\pi) - 1)$$

$$a_n = \begin{cases} -\dfrac{2}{(n\pi)^2}, & n = 1, 3, 5, \ldots \\ 0, & n = 2, 4, 6, \ldots \end{cases}$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)dt$$

$$= \frac{2}{2}\left(\int_0^1 t\sin(n\pi t)dt + \int_1^2 0\sin(n\pi t)dt\right)$$

$$= \int_0^1 t\sin(n\pi t)dt + 0 = \left[-t\frac{\cos(n\pi t)}{n\pi}\right]_0^1 + \left[\frac{\sin(n\pi t)}{(n\pi)^2}\right]_0^1 = -\frac{1}{n\pi}\cos(n\pi)$$

$$b_n = \begin{cases} \dfrac{1}{n\pi}, & n = 1, 3, 5, \ldots \\ -\dfrac{1}{n\pi}, & n = 2, 4, 6, \ldots \end{cases}$$

Substituting the values of a_0, a_n and b_n, gives

$$x(t) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

$$= \frac{1}{4} - \frac{2}{\pi^2}\cos(\pi t) - \frac{2}{(3\pi)^2}\cos(3\pi t) - \frac{2}{(5\pi)^2}\cos(5\pi t) - ....$$

$$+ \frac{1}{\pi}\sin(\pi t) - \frac{1}{2\pi}\sin(2\pi t) + \frac{1}{3\pi}\sin(3\pi t) - ...$$

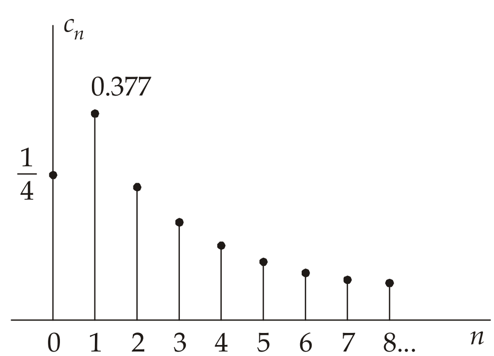
The line spectrum is shown in figure below. The even harmonic amplitudes are given directly by $|b_n|$ since there are no even-harmonic cosine terms. However, the odd-harmonic amplitudes must be computed using $c_n = \sqrt{a_n^2 + b_n^2}, n \geq 1$ and $c_0 = a_0$. Thus,

$$c_1 = \sqrt{\left(\frac{2}{\pi^2}\right)^2 + \left(\frac{1}{\pi}\right)^2} = 0.377$$

$$C_2 = \frac{1}{2\pi} = 0.159$$

$$c_3 = 0.109$$

$$c_5 = 0.064$$



Q.6 (b) (ii) Solution:

Given that, $$x[n] = \left(\frac{4}{5}\right)^n u(n)$$

So, $$X(e^{j\omega}) = \frac{1}{1 - \frac{4}{5}e^{-j\omega}}$$

and $$y[n] = n \cdot \left(\frac{4}{5}\right)^n u[n] \xleftrightarrow{\text{DTFT}} j\frac{d}{d\omega}\left[\frac{1}{1 - \frac{4}{5}e^{-j\omega}}\right]$$

$$Y(e^{j\omega}) = j\left[\frac{-j\frac{4}{5}e^{-j\omega}}{\left(1-\frac{4}{5}e^{-j\omega}\right)^2}\right] = \frac{\frac{4}{5}e^{-j\omega}}{\left(1-\frac{4}{5}e^{-j\omega}\right)^2}$$

So,
$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\frac{4}{5}e^{-j\omega}}{1-\frac{4}{5}e^{-j\omega}}$$

Q.6 (c) (i) Solution:

Consider the given difference equation

$$y(n) = 0.5y(n-1) + x(n)$$

Taking its z-transform, we obtain

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{1-0.5z^{-1}}$$

This system has a pole at z = 0.5. Taking the z-transform of the input,

$$X(z) = \frac{10(1-z^{-1}\cos(\pi/4)}{1-2z^{-1}\cos(\pi/4)+z^{-2}} = \frac{10\left[1-\frac{1}{\sqrt{2}}z^{-1}\right]}{1-\sqrt{2}z^{-1}+z^{-2}}$$

The output of the system is given

$$Y(z) = H(z)X(z) = \frac{10\left[1-(1/\sqrt{2})z^{-1}\right]}{(1-0.5z^{-1})(1-\sqrt{2}z^{-1}+z^{-2})}$$

$$= \frac{10\left[1-(1/\sqrt{2})z^{-1}\right]}{(1-0.5z^{-1})(1-e^{j\pi/4}z^{-1})(1-e^{-j\pi/4}z^{-1})}$$

Using partial fraction expansion, we obtain

$$Y(z) = \underbrace{\frac{6.3}{1-0.5z^{-1}}}_{\text{pole of the system}} + \underbrace{\frac{6.78e^{-j28.7}}{1-e^{j\pi/4}z^{-1}} + \frac{6.78e^{j28.7}}{1-e^{-j\pi/4}z^{-1}}}_{\text{poles of the forcing function, i.e., input}}$$

The inverse z-transform of this equation yields

$$y(n) = \underbrace{6.3(0.5)^n u(n)}_{\text{transient response}} + \underbrace{\left[6.78e^{-j28.7}e^{jn\pi/4} + 6.78e^{j28.7}e^{-jn\pi/4}\right]u(n)}_{\text{steady-state response}}$$

The system is initially relaxed, i.e., all the initial conditions are zero. Therefore, the natural or transient response is

$$y_n(n) = 6.3(0.5)^n u(n)$$

and the forced or steady-state response is

$$y_f(n) = \left[6.78e^{-j28.7}e^{j\frac{n\pi}{4}} + 6.78e^{-j28.7}e^{-j\frac{n\pi}{4}}\right]u(n)$$

$$= 13.56\cos\left(\frac{\pi}{4}n - 28.7°\right)u(n)$$

Q.6 (c) (ii) Solution:

Given $y[n] = x[n] + by[n-1]$ ...(i)

Initial condition $y[-1] = P$ ...(ii)

Taking unilateral z-transform on both the sides of equation (i) we get

$$Y(z) = X(z) + b\left[z^{-1}Y(z) + y[-1]\right]$$

$$Y(z) = X(z) + bz^{-1}Y(z) + by[-1]$$

$$(1 - bz^{-1})Y(z) = X(z) + bP \quad \text{...(iii)}$$

and $x[n] = e^{j\omega n}u[n]$

$$X(z) = \frac{1}{\left(1 - e^{j\omega_0}z^{-1}\right)} \quad \text{...(iv)}$$

$$\therefore (1 - bz^{-1})Y(z) = \frac{1}{(1 - e^{j\omega_0}z^{-1})} + bP$$

$$Y(z) = \frac{1}{(1 - e^{j\omega_0}z^{-1})(1 - bz^{-1})} + \frac{bP}{(1 - bz^{-1})}$$

Taking inverse z-transform we have

$$y[n] = ZT^{-1}\left[\frac{1}{(1 - e^{j\omega_0}z^{-1})(1 - bz^{-1})} + \frac{bP}{1 - bz^{-1}}\right]$$

According to linearity property of z-transform

$$y[n] = ZT^{-1}\left[\frac{1}{(1 - e^{j\omega_0}z^{-1})(1 - bz^{-1})}\right] + ZT^{-1}\left[\frac{bP}{(1 - bz^{-1})}\right]$$

$$y[n] = ZT^{-1}\left[\frac{1}{(1 - b/e^{j\omega_0})(1 - e^{j\omega_0}z^{-1})} + \frac{1}{(1 - e^{j\omega_0}/b)(1 - bz^{-1})}\right] + bP\cdot(b)^n u[n]$$

$$y[n] = ZT^{-1}\frac{1}{(1 - b/e^{j\omega_0})(1 - e^{j\omega_0}z^{-1})} + ZT^{-1}\frac{1}{(1 - e^{j\omega_0}/b)(1 - bz^{-1})} + b^{n+1}Pu[n]$$

$$y[n] = \frac{1}{(1-b/e^{j\omega_0})}(e^{j\omega_0})^n\, u[n] + \frac{1}{(1-e^{j\omega_0}/b)} b^n\, u[n] + b^{n+1}\, Pu[n]$$

$$y[n] = \left[\frac{e^{j\omega_0 n}}{(1-b/e^{j\omega_0})} + \frac{b^n}{(1-e^{j\omega_0}/b)} + b^{n+1}P\right] u[n]$$

Q.7 (a) Solution:

(i) Given that,

$$x_1(t) = e^{-2t}\, u(t)$$

$$|x_1(t)|^2 = e^{-4t}\, u(t)$$

Therefore,

$$E_x = \int_{-\infty}^{\infty} \left|x_1(t)^2 dt\right| = \int_0^{\infty} e^{-4t} dt = \frac{1}{4}$$

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x_1(t)|^2\, dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} e^{-4t} u(t) dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} e^{-4t} dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\frac{e^{-4t}}{-4}\Bigg|_0^{T/2} = \lim_{T\to\infty}\frac{1}{T}\left[e^{-4T}-1\right] = 0$$

Since, $P_x = 0$ and $0 < E_x < \infty$, $x_1(t)$ is an energy signal

(ii) Given that,

$$x_2(t) = e^{j(2t + \pi/4)}$$

$$|x_2(t)|^2 = 1$$

Therefore,

$$E_a = \int_{-\infty}^{\infty} (1) dt = \infty$$

$$P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} |x_2(t)|^2\, dt$$

$$= \lim_{T\to\infty}\int_{-T/2}^{T/2} dt = \lim_{T\to\infty}\frac{1}{T}T = 1$$

Since, $E_x = \infty$ and $0 < P_x < \infty$, $x_2(t)$ is a power signal

(iii) Given that,

$$x_3(n) = \cos\left(\frac{\pi}{4}n\right)$$

$$|x_3(n)|^2 = \cos^2\left(\frac{\pi}{4}n\right) = \frac{1+\cos\left(\left(\frac{\pi}{2}\right)n\right)}{2}$$

Therefore,
$$E_x = \sum_{n=-\infty}^{\infty} |x_3(n)|^2 = \sum_{n=-\infty}^{\infty} \frac{1+\cos\left(\left(\frac{\pi}{2}\right)n\right)}{2}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2} + \sum_{n=-\infty}^{\infty} \frac{\cos\left(\left(\frac{\pi}{2}\right)n\right)}{2} = \infty$$

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_3(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1+\cos\left(\frac{\pi}{2}\right)n}{2}$$

$$= \left(\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1}{2}\right) + \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1}{2}\cos\left(\frac{\pi}{2n}\right)$$

$$= \left(\lim_{N\to\infty} \frac{1}{2}\frac{1}{(2N+1)}(2N+1)\right) + 0 = \frac{1}{2}$$

Since $E_x = \infty$ and $0 < P_x < \infty$, $x_3(n)$ is a power signal.

Q.7 (b) Solution:

Using the partial fraction expansion, we obtain

$$X(s) = \frac{1}{s+1} - \frac{2}{s-1} + \frac{1}{s+2}$$

$X(s)$ has poles at –1, 1, and –2.

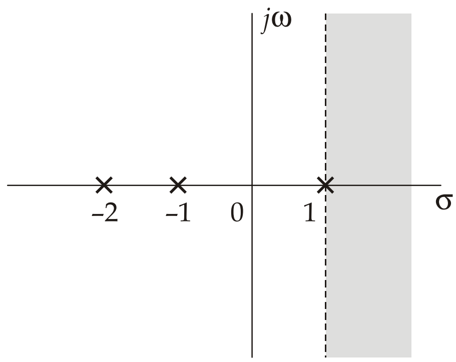
(i) The ROC and the locations of the poles are depicted in figure (a). The ROC, $\Re\{s\} > 1$, is to the right of the rightmost pole, so all the three poles correspond to causal (right-sided) signals. Therefore,

$$X(s) = \underbrace{\frac{1}{s+1}}_{\text{R.H.S.}} - \underbrace{\frac{2}{s-1}}_{\text{R.H.S.}} + \underbrace{\frac{1}{s+2}}_{\text{R.H.S.}}$$

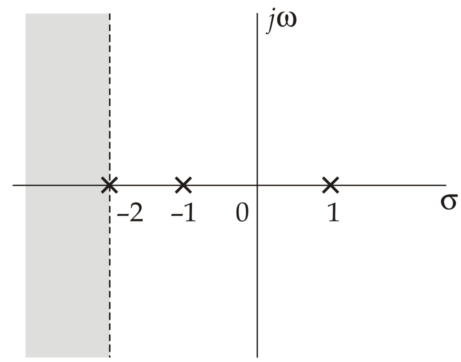
For ROC > 1;

Here, $$x(t) = \left(e^{-t} - 2e^{t} + e^{-2t}\right)u(t)$$

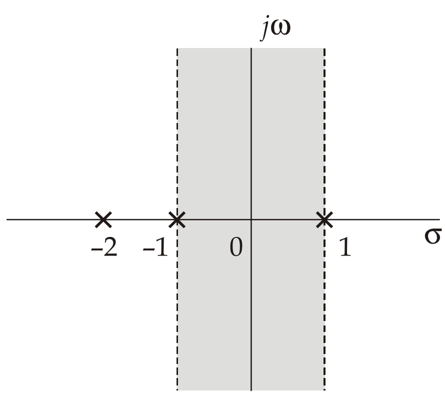
$$e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2}$$



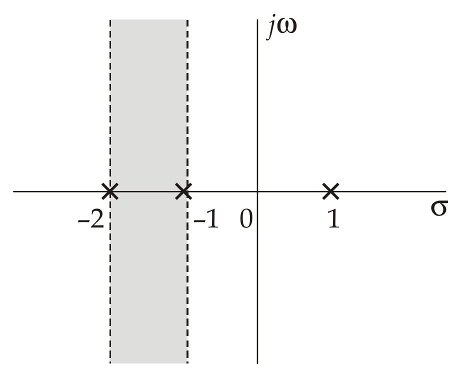
(a)



(b)



(c)



(d)

and hence

$$x(t) = e^{-t}u(t) - 2e^{t}u(t) + e^{-2t}u(t)$$

(ii) The ROC and the locations of the poles are depicted in fig. (b). The ROC $\Re\{s\} < -2$, is to the left of the leftmost pole, so all the three poles correspond to anticausal (left-sided) signals. Therefore,

$$-e^{-t}u(-t) \longleftrightarrow \frac{1}{s+1}$$

$$-2e^{t}u(-t) \longleftrightarrow \frac{2}{s-1}$$

$$-e^{-2t}u(-t) \longleftrightarrow \frac{1}{s+2}$$

and hence, $x(t) = -e^{-t}u(-t) + 2e^{t}u(-t) - e^{-2t}u(-t)$

(iii) The ROC and the locations of the poles are depicted in figure (c). The ROC $-1 < \Re\{s\} < 1,$ is a strip. The pole of the first term is at –1. The ROC lies to the right of this pole, so this pole corresponds to a causal (right-sided) signal. Therefore,

$$e^{-t}u(t) \longleftrightarrow \frac{1}{s+1}$$

The second term has a pole at s = 1. Here the ROC is to the left of this pole, so this pole corresponds to an anticausal (left-sided) signal. Therefore,

$$-2e^{t}u(-t) \longleftrightarrow \frac{2}{s-1}$$

The third term has a pole at s = –2. Here the ROC is to the right of this pole, so this pole corresponds to a causal (right-sided) signal. Therefore,

$$e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2}$$

and hence we obtain

$$x(t) = e^{-t}u(t) + 2e^{t}u(-t) + e^{-2t}u(t)$$

(iv) The ROC and the locations of the poles are depicted in figure (c). The ROC, $-2 < \Re\{s\} < -1,$ is a strip. The pole of the first term is at –1. The ROC lies to the left of this pole, so this pole corresponds to an anticausal (left-sided) signal. Therefore,

$$-e^{-t}u(-t) \longleftrightarrow \frac{1}{(s+1)}$$

The second term has a pole at s = 1. Here the ROC is to the left of this pole, so this pole corresponds to an anti-causal (left-sided) signal. Therefore,

$$-2e^{t}u(-t) \longleftrightarrow \frac{2}{s-1}$$

The third term has a pole at s = –2. Here the ROC is to the right of this pole, so this pole corresponds to causal (right-sided) signal. Therefore,

$$e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2}$$

and hence we obtain $x(t) = -e^{-t}u(-t) + 2e^{t}u(-t) + e^{-2t}u(t)$

Q.7 (c) Solution:

(i) Determine the Discrete-time Transfer Function $H(z)$

The Bilinear Transformation maps the s-plane to the z-plane using the substitution:

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

Given, $T = 0.2$ seconds, the substitution factor is $\frac{2}{0.2} = 10$

$$s = 10\left(\frac{z-1}{z+1}\right)$$

Substitute s into $H_a(s)$:

$$H(z) = \frac{10}{\left[10\left(\frac{z-1}{z+1}\right)\right]^2 + 7\left[10\left(\frac{z-1}{z+1}\right)\right] + 10}$$

Divide numerator and denominator by 10 to simplify:

$$H(z) = \frac{(z+1)^2}{10(z-1)^2 + 7(z-1)(z+1) + (z+1)^2}$$

$$= \frac{z^2+2z+1}{10(z^2-2z+1)+7(z^2-1)+(z^2+2z+1)}$$

$$= \frac{z^2+2z+1}{10z^2-20z+10+7z^2-7+z^2+2z+1}$$

$$H(z) = \frac{z^2+2z+1}{18z^2-18z+4}$$

(ii) Difference Equation:

To find the difference equation, express $H(z)$ in terms of z^{-1} :

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+2z^{-1}+z^{-2}}{18-18z^{-1}+4z^{-2}}$$

Cross-multiplying:

$$Y(z)(18 - 18z^{-1} + 4z^{-2}) = X(z)(1 + 2z^{-1} + z^{-2})$$

Taking the Inverse Z-Transform:

$$18y[n] - 18y[n-1] + 4y[n-2] = x[n] + 2x[n-1] + x(n-2)$$

(iii) Poles and Stability:

The poles are the roots of the denominator

$$18z^2 - 18z + 4 = 0$$

Dividing by 2:

$$9z^2 - 9z + 2 = 0$$

Using the quadratic formula,

$$z = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(9)(2)}}{2(9)} = \frac{9 \pm \sqrt{81-72}}{18}$$

$$z = \frac{9 \pm 3}{18}$$

- Pole 1 (z_1) : $\frac{12}{18} = 0.667$
- Pole 2 (z_2) : $\frac{6}{18} = 0.333$

Stability comment: Since both poles $|z_1| < 1$ and $|z_2| < 1$ lie strictly inside the unit circle, the digital filter is stable.

Q.8 (a) Solution:

(i) By definition,

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} (0.8)^k u(k)(0.4)^{n-k} u(n-k)$$

The power limit on the convolution sum simplifies to $k = 0$ [because $u(k) = 0$, $k < 0$], the upper limit to $k = n$, and we get

$$y(n) = \sum_{k=0}^{n} (0.8)^k (0.4)^{n-k} = (0.4)^n \sum_{k=0}^{n} 2^k = (0.4)^n \frac{1-2^{n+1}}{1-2}$$

$$= (0.4)^n (2^{n+1} - 1)$$

$$= \left[2(0.8)^n - (0.4)^n\right] u(n)$$

(ii) Given that,

$$x(n) = u(n-1) = \begin{cases} 1, & n \geq 1 \\ 0, & n < 1 \end{cases}$$

and $$h(n) = \alpha^n u(n-1) = \begin{cases} \alpha^n, & n \geq 1 \\ 0, & n < 1 \end{cases}$$

By definition, $$y(n) = x(n) * h(n) = \sum_{k=k_1}^{n-k_2} x(k)h(n-k)$$

where $k_1 = 1$ and $k_2 = 1$

$$y(n) = 0$$
$$n < (k_1 + k_2) \rightarrow n < 2$$

and $$y(n) = \sum_{k=k_1}^{n-k_2} x(k)h(n-k) \qquad n \geq (k_1 + k_2)$$

$$= \sum_{k=1}^{n-1} \alpha^{n-k} \qquad n \geq 1 + 1 = 2$$

$$= \alpha^n \sum_{k=1}^{n-1} \alpha^{-k} = \alpha^n \sum_{m=0}^{n-2} \alpha^{-(m+1)}$$

$$= \alpha^n \alpha^{-1} \sum_{m=0}^{n-2} \alpha^{-(m)} = \frac{\alpha - \alpha^n}{1-\alpha}$$

$$y(n) = \begin{cases} \dfrac{\alpha - \alpha^n}{1-\alpha}, & n \geq 2 \\ 0, & n < 0 \end{cases}$$

or, more compactly $$y(n) = \frac{\alpha - \alpha^n}{1-\alpha} u(n-2)$$

(iii) Given that,

$$x(n) = nu\,(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

and $$h(n) = a^{-n} u(n-1) = \begin{cases} a^{-n}, & n \geq 1 \\ 0, & n < 1 \end{cases}$$

By definition $$y(n) = x(n) * h(n) = \sum_{k=k_1}^{n-k_2} x(k)h(n-k)$$

where $k_1 = 0$ and $k_2 = 1$

$$y(n) = 0, \qquad n < (k_1 + k_2) \rightarrow n < 1$$

and $$y(n) = \sum_{k=k_1}^{n-k_2} x(k)h(n-k)$$

$n \geq (k_1 + k_2)$

$$= \sum_{k=0}^{n-1} ka^{-(n-k)} \qquad n \geq (0+1) = 1$$

$$= a^{-n}\sum_{k=0}^{n-1} ka^k = \frac{a^{-n+1}}{(1-a)^2}\left[1 - na^{n-1} + (n-1)a^n\right]$$

Therefore, $$y(n) = \begin{cases} \dfrac{a^{-n+1}}{(1-a)^2}\left[1 - na^{n-1} + (n-1)a^n\right], & n \geq 1 \\ 0, & n < 1 \end{cases}$$

or, more compactly, $$y(n) = \frac{a^{-n+1}}{(1-a)^2}\left[1 - na^{n-1} + (n-1)a^n\right]u(n-1)$$

Q.8 (b) Solution:

(i) Trigonometric Fourier Series Coefficients:

The square wave is an even signal, centered at $t = 0$, with a period T and fundamental frequency,

$$\omega_0 = \frac{2\pi}{T}$$

1. DC Coefficient (a_0):

   The average value over one period:

$$a_0 = \frac{1}{T}\int_{-T/4}^{T/4} A\,dt = \frac{A}{T}\left[\frac{T}{2}\right] = \frac{A}{2}$$

2. Cosine Coefficients (a_n):

$$a_n = \frac{2}{T}\int_{-T/4}^{T/4} A\cos(n\omega_0 t)\,dt$$

$$= \frac{2A}{T}\left[\frac{\sin(n\omega_0 t)}{n\omega_0}\right]_{-T/4}^{T/4} = \frac{2A}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

- For n even :

$$a_n = 0$$

- For $n = 1, 5, 9...$:

$$a_n = \frac{2A}{n\pi}$$

- For $n = 3, 7, 11...$:

$$a_n = -\frac{2A}{n\pi}$$

3. Sine Coefficients (b_n)

   Since $x(t)$ is an even function,

$$b_n = 0 \text{ for all } n$$

(ii) Gibbs Phenomenon:

The Gibbs Phenomenon describes the behavior of the Fourier Series at a point of discontinuity.

- Observation: When approximating a square wave with N harmonics, an overshoot occurs at the edges of the pulse.
- The 9% Rule: As $N \to \infty$ the frequency of the oscillations increases, but the amplitude of the overshoot does not go to zero. It settles at approximately 8.95% of the total jump height.
- Significance: It proves that the Fourier series converges to the midpoint of the jump at the point of discontinuity, but produces "ringing" artifacts in the vicinity.

(iii) Duality Property Application:

Duality property: If $x(t) \leftrightarrow X(\omega)$, then $X(t) \leftrightarrow 2\pi x(-\omega)$.

1. Standard Pair: We know that a rectangular pulse $x(t) = \text{rect}\left(\frac{t}{2a}\right)$ has the Fourier transform $X(\omega) = \frac{2\sin(a\omega)}{\omega}$.

2. Apply Duality: Swap the variables t and ω

   If $$x(t) = \frac{2\sin(at)}{t}$$

   then $$X(\omega) = 2\pi \cdot \text{rect}\left(\frac{-\omega}{2a}\right)$$

3. Adjust for Constants: Divide both sides by 2π

$$F\left\{\frac{\sin(at)}{\pi t}\right\} = \text{rect}\left(\frac{\omega}{2a}\right)$$

4. Final Result:

   The Fourier transform $G(\omega)$ is a Rectangular Function with a height of 1 and a width of $2a$ (from $-a$ to a).

Q.8 (c) Solution:

(i) 1. We may redraw the block diagram as shown in Figure (1).

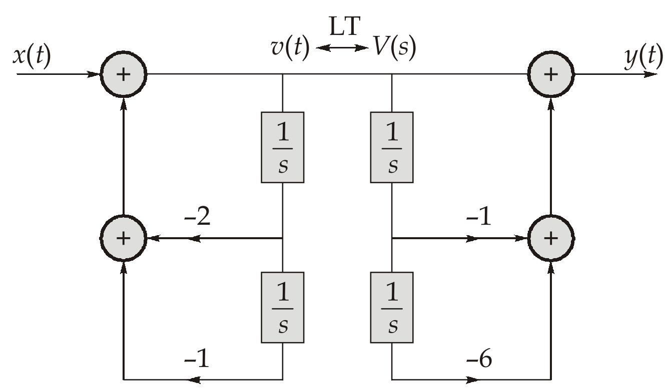


Figure (1)

Let $$H_1(s) = \frac{V(s)}{X(s)} \text{ and } H_2(s) = \frac{Y(s)}{V(s)}$$

$\therefore$ $$H(s) = H_1(s)\cdot H_2(s) \quad \text{...(i)}$$

Now, consider the left half section of the Figure (1).

$$V(s) = X(s)+\frac{1}{s}V(s)(-2)+(-1)\frac{V(s)}{s^2}$$

$$V(s)\left[1+\frac{2}{s}+\frac{1}{s^2}\right] = X(s)$$

$$\frac{V(s)}{X(s)} = \frac{s^2}{(s^2+2s+1)}$$

$$\frac{V(s)}{X(s)} = H_1(s) = \frac{s^2}{s^2+2s+1} \quad \text{...(ii)}$$

Now, from the right half section of Figure (1).

$$Y(s) = V(s)+\frac{1}{s}V(s)(-1)+\frac{1}{s^2}V(s)(-6)$$

$$Y(s) = V(s)\left[1-\frac{1}{s}-\frac{6}{s^2}\right]$$

$$\frac{Y(s)}{V(s)} = H_2(s) = \frac{(s^2-s-6)}{s^2} \quad \text{...(iii)}$$

Now, from equation (i), (ii) and (iii), we get

$$H(s) = \frac{s^2}{(s^2+2s+1)} \times \frac{(s^2-s-6)}{s^2}$$

$$H(s) = \frac{(s^2-s-6)}{(s^2+2s+1)} = \frac{Y(s)}{X(s)}$$

$$Y(s)[s^2 + 2s + 1] = X(s)[s^2 - s - 6]$$

The inverse Laplace transform of the above equation yields the differential equation of the system relating $x(t)$ and $y(t)$.

$$\frac{d^2y(t)}{dt^2} + \frac{2dy(t)}{dt} + y(t) = \frac{d^2x(t)}{dt^2} - \frac{dx(t)}{dt} - 6x(t)$$

2.

$$H(s) = \frac{(s^2-s-6)}{(s+1)^2} = \frac{(s-3)(s+2)}{(s+1)^2}$$

We know that for a system to be stable, all the poles must lie in the left half of the s-plane. The two poles of the system are at $s = -1$. We may conclude that this system is stable.

(ii) Given differential equation is

$$\frac{d^2y(t)}{dt^2} + \frac{6dy(t)}{dt} + 8y(t) = 2x(t) \qquad \text{...(i)}$$

Taking Laplace transform of equation (i), we get

$$s^2Y(s) + 6sY(s) + 8Y(s) = 2X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{2}{s^2+6s+8}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2}{(s+2)(s+4)}$$

When input $x(t) = te^{-2t}u(t)$

$$X(s) = \frac{1}{(s+2)^2}$$

since,

$$Y(s) = X(s) \,.\, H(s) = \frac{1}{(s+2)^2} \cdot \frac{2}{(s+2)(s+4)}$$

$$Y(s) = \frac{2}{(s+2)^3(s+4)}$$

Breaking into partial fractions

$$\therefore \qquad Y(s) = \frac{A}{s+4} + \left[\frac{B_3}{(s+2)} + \frac{B_2}{(s+2)^2} + \frac{B_1}{(s+2)^3}\right]$$

$$\therefore \qquad A = \left.\frac{2}{(s+2)^3}\right|_{s=-4} = \frac{2}{-8} = \frac{-1}{4}$$

and

$$B_3 = \frac{1}{\lfloor 2}\frac{d^2}{ds^2}\left[(s+2)^3 \cdot \frac{2}{(s+2)^3(s+4)}\right]_{s=-2}$$

$$= \frac{1\times 2}{2}\left[\frac{d^2}{ds^2}\left(\frac{1}{s+4}\right)\right]_{s=-2}$$

$$= \frac{d}{ds}\left\{\frac{-1}{(s+4)^2}\right\}_{s=-2} = \left.\frac{2(s+4)}{(s+4)^4}\right|_{s=-2}$$

$$B_3 = \frac{1}{4}$$

also,

$$B_2 = \frac{1}{\lfloor 1}\frac{d}{ds}\left[(s+2)^3 \cdot \frac{2}{(s+2)^3(s+4)}\right]_{s=-2} = -\frac{1}{2}$$

and

$$B_1 = (s+2)^3 \cdot \left.\frac{2}{(s+2)^3(s+4)}\right|_{s=-2} = 1$$

$$\therefore \qquad Y(s) = \left(-\frac{1}{4}\right)\frac{1}{(s+4)} + \frac{1/4}{(s+2)} + \left(\frac{-1}{2}\right)\frac{1}{(s+2)^2} + \frac{1}{(s+2)^3}$$

$$\Rightarrow \qquad y(t) = \left(-\frac{1}{4}e^{-4t} + \left(\frac{e^{-2t}}{4}\right) - \frac{1}{2}te^{-2t} + \frac{1}{2}t^2e^{-2t}\right)u(t)$$

$$= \left[e^{-2t}(0.25 - 0.5t + 0.5t^2) - 0.25e^{-4t}\right]u(t)$$

❍❍❍❍