Detailed Solutions

ESE-2025 Mains Test Series

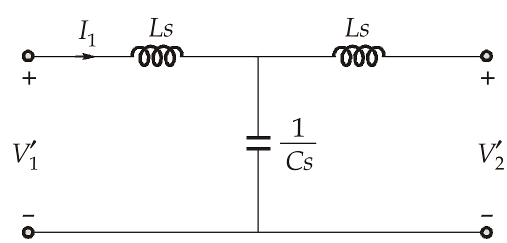
Electrical Engineering Test No : 1

Section A : Electric Circuits

Q.1 (a) Solution:

The above network can be considered as a series connection of two networks, N_1 and N_2.

For the network N_1



Applying KVL to mesh 1,

$$V_1' = \left(Ls + \frac{1}{Cs}\right)I_1 + \left(\frac{1}{Cs}\right)I_2 \quad \text{...(i)}$$

Applying KVL to mesh 2,

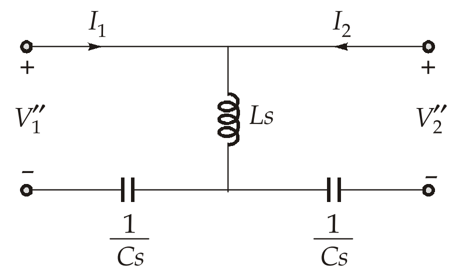
$$V_2' = \left(\frac{1}{Cs}\right)I_1 + \left(Ls + \frac{1}{Cs}\right)I_2 \quad \text{...(ii)}$$

Comparing equations (i) and (ii) with Z-parameters equation,

We get,

$$\begin{bmatrix} Z_{11}' & Z_{12}' \\ Z_{21}' & Z_{22}' \end{bmatrix} = \begin{bmatrix} Ls + \frac{1}{Cs} & \frac{1}{Cs} \\ \frac{1}{Cs} & Ls + \frac{1}{Cs} \end{bmatrix}$$

For the network N_2



Applying KVL to mesh 1,

$$V_1'' = \left(Ls + \frac{1}{Cs}\right)I_1 + (Ls)I_2 \quad ...(i)$$

Applying KVL to mesh 2,

$$V_2'' = (Ls)I_1 + \left(Ls + \frac{1}{Cs}\right)I_2 \quad ...(ii)$$

Comparing equations (i) and (ii) with Z-parameters equations, we get

$$\begin{bmatrix} Z_{11}'' & Z_{12}'' \\ Z_{21}'' & Z_{22}'' \end{bmatrix} = \begin{bmatrix} Ls + \frac{1}{Cs} & Ls \\ Ls & Ls + \frac{1}{Cs} \end{bmatrix}$$

Hence, the overall Z-parameters of the network are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_{11}' + Z_{11}'' & Z_{12}' + Z_{12}'' \\ Z_{21}' + Z_{21}'' & Z_{22}' + Z_{22}'' \end{bmatrix}$$

$$= \begin{bmatrix} 2Ls + \frac{2}{Cs} & Ls + \frac{1}{Cs} \\ Ls + \frac{1}{Cs} & 2Ls + \frac{2}{Cs} \end{bmatrix}$$

$$= \left(Ls + \frac{1}{Cs}\right)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Q.1 (b) Solution:

Applying Kirchoff current law at node A,

We have,

$$I - \frac{V_A}{j\omega L} - \frac{V_A}{R} - \frac{V_A + 10^4 I_1}{1/j\omega C} = 0$$

and $$I_1 = \frac{V_A}{R}$$

Substituting this we have

$$I = \frac{V_A}{j\omega L} + \frac{V_A}{R} + \frac{V_A + \dfrac{10^4 V_A}{R}}{\dfrac{1}{j\omega C}}$$

$$= V_A\left[\frac{1}{R} + j\left\{\omega C\left(1 + \frac{10^4}{R}\right) - \frac{1}{\omega L}\right\}\right]$$

$$Y_{in} = \frac{I}{V_A} = \frac{1}{R} + j\left[\omega C\left(1 + \frac{10^4}{1.5\times10^4} - \frac{1}{5\times10^{-3}\omega}\right)\right]$$

For resonance,

$$\omega_0 C(1 + 0.667) = \frac{10^3}{5\omega_0}$$

$$\omega_0 \times 10 \times 10^{-6}(0.667) = \frac{10^3}{5\omega_0}$$

$$\omega_0^2 = \frac{10^3\times10^5}{1.667\times5} = 11.99\times10^6$$

$$\omega_0 = 3.46 \text{ K rad/sec}$$

$$f_0 = 551 \text{ Hz}$$

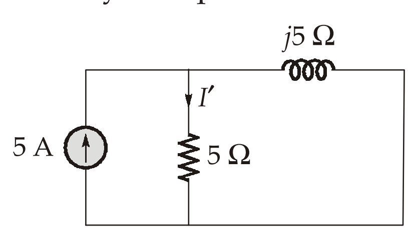
The quality factor, $$\frac{R}{\omega_0 L} = \frac{15\times10^3}{3.46\times10^3\times5\times10^{-3}} = 867$$

Hence, Bandwidth, $$\frac{f_0}{Q_0} = \frac{551}{867} = 0.635 \text{ Hz}$$

Q.1 (c) (i) Solution:

Let the current flowing through 5 Ω resistor is I' when 5 A current source acting alone,

Replacing the 10 V voltage source by its equivalent source impedance,

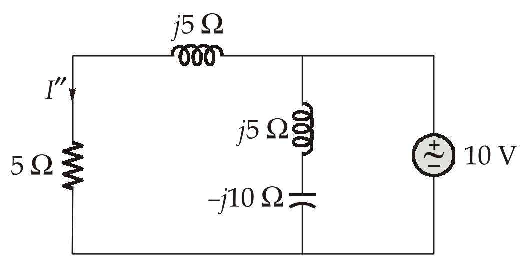


Using current division rule,

$$I' = \frac{j5}{5+j5} \times 5\angle 0° = (2.5 + j2.5)\text{ A}$$

Let the current through 5 Ω resistor is I'' when 10 V voltage source acting alone,

Replacing the 5 A source by its equivalent source impedance,



$$I'' = \frac{10}{5+j5} = (1 - j1)\text{A}$$

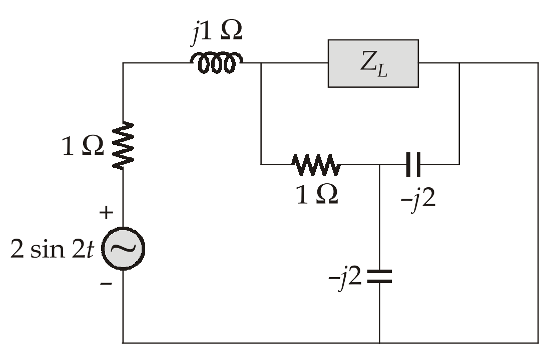
Now, current $I = I' + I''$

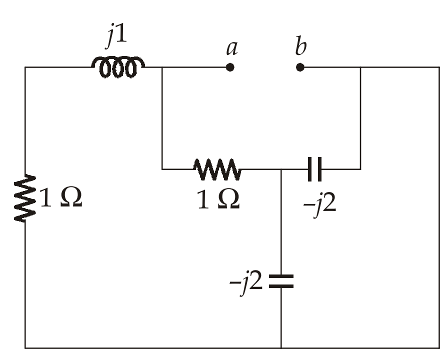
$$I = (2.5 + j2.5) + (1 - j1)$$

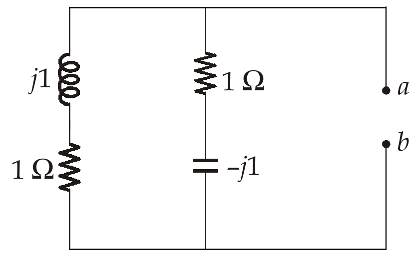
$$= (3.5 + j1.5)\text{ A}$$

Q.1 (c) (ii) Solution:

Since ω = 2 rad/sec, the network is drawn as







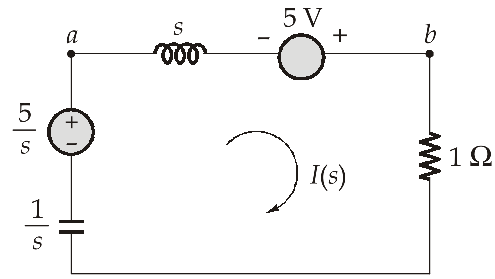
$$Z_L = \frac{(1+j1)(1-j1)}{2} = \frac{1+1}{2} = 1\,\Omega$$

Hence for maximum power through Z_L, it should be $Z_L = 1\,\Omega$.

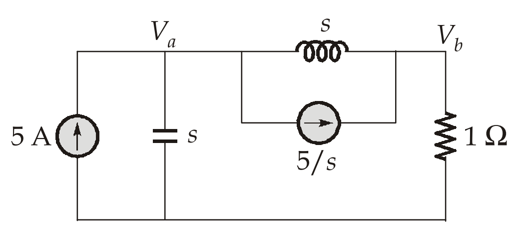
Q.1 (d) Solution:

Under steady state condition $i_L(0^-) = \frac{5}{1} = 5\text{ A}$ and $V_C(0^-) = 5$ Volt.

The transform network becomes



Transforming the voltage sources into current sources we have the network



Writing nodal equations at node a and b respectively we have,

$$\begin{bmatrix} \left(s+\frac{1}{s}\right) & -\frac{1}{s} \\ -\frac{1}{s} & 1+\frac{1}{s} \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} 5-\frac{5}{s} \\ \frac{5}{s} \end{bmatrix}$$

Using Cramer's rule we have

$$V_a = \frac{\begin{vmatrix} 5-\frac{5}{s} & -\frac{1}{s} \\ \frac{5}{s} & 1+\frac{1}{s} \end{vmatrix}}{\begin{vmatrix} s+\frac{1}{s} & -\frac{1}{s} \\ -\frac{1}{s} & 1+\frac{1}{s} \end{vmatrix}} = \frac{\left(5-\frac{5}{s}\right)\left(1+\frac{1}{s}\right)+\frac{5}{s^2}}{\left(s+\frac{1}{s}\right)\left(1+\frac{1}{s}\right)-\frac{1}{s^2}}$$

$$= \frac{5s^2}{s^3+s^2+s} = \frac{5s+\frac{5}{2}-\frac{5}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{5\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{\frac{5}{2}\times\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$V_a(t) = 5e^{-t/2}\cos\frac{\sqrt{3}}{2}t - \frac{5}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\cdot e^{-t/2}$$

$$= 5e^{-t/2}\left[\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right]\text{volts}$$

Similarly, $$V_b(s) = \frac{\begin{vmatrix} s+\frac{1}{s} & 5-\frac{5}{s} \\ -\frac{1}{s} & \frac{5}{s} \end{vmatrix}}{\frac{(s^3+s^2+s)}{s^2}} = \frac{5(s+1)}{s^2+s+1} = 5\left[\frac{s+\frac{1}{2}-\frac{1}{2}}{s^2+s+1}+\frac{1}{s^2+s+1}\right]$$

$$= 5\left[\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\frac{1}{2}\times\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$V_b(t) = 5e^{-t/2}\cos\frac{\sqrt{3}}{2}t + \frac{5}{\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}}{2}t$$

$$= 5e^{-t/2}\left[\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right]\text{volts}$$

Q.1 (e) Solution:

$$v_s(t) = 2\cos t$$

Angular frequency, $\omega = 1$ rad/sec.

$$\therefore \quad X_L = \omega L = 1 \times 1 = 1\ \Omega$$

$$X_C = \frac{1}{\omega C} = \frac{1}{1\times 1} = 1\Omega$$

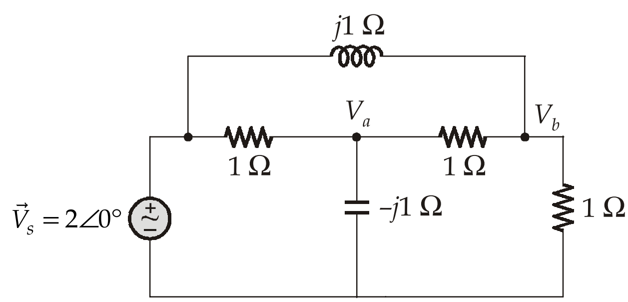
$$\vec{V}_s = 2\angle 0°$$

$$\omega = 1 \text{ rad/sec,}$$

$$X_L = 1\ \Omega$$

$$X_C = \frac{1}{|X|} = 1\ \Omega$$

Now the circuit in frequency domain



By KCL at node - a

$$\frac{V_a - 2}{1} + \frac{V_a}{-j1} + \frac{V_a - V_b}{1} = 0$$

$$\Rightarrow \quad (2 + j)V_a - V_b = 2 \quad \text{...(i)}$$

By KCL at node - b

$$\frac{V_b - 2}{j1} + \frac{V_b - V_a}{1} + \frac{V_b}{1} = 0$$

$$\Rightarrow \quad -V_a + (2 - j)V_b = -j2 \quad \text{...(ii)}$$

From equation (i) and (ii),

$$\begin{bmatrix} 2+j & -1 \\ -1 & 2-j \end{bmatrix}\begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} 2 \\ -j2 \end{bmatrix}$$

$$V_a = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} 2 & -1 \\ -j2 & 2-j \end{vmatrix}}{\begin{vmatrix} 2+j & -1 \\ -1 & 2-j \end{vmatrix}} = \frac{4\sqrt{2}\angle -45°}{4} = \sqrt{2}\angle -45° \text{ Volts}$$

Similarly,
$$V_b = \frac{\begin{vmatrix} 2+j & 2 \\ -1 & -j2 \end{vmatrix}}{\begin{vmatrix} 2+j & -1 \\ -1 & 2-j \end{vmatrix}} = \frac{4\sqrt{2}\angle -45°}{4} = \sqrt{2}\angle -45° \text{ Volts}$$

Now,
$$\vec{I}_a = \frac{V_s - V_a}{1} = \frac{2\angle 0° - \sqrt{2}\angle -45°}{1} = \sqrt{2}\angle 45° \text{ A}$$

$$\vec{I}_b = \frac{V_b}{1} = \sqrt{2}\angle -45° \text{ A}$$

Therefore,
$$i_a(t) = \sqrt{2}\cos(t + 45°)\text{A}$$
$$i_b(t) = \sqrt{2}\cos(t - 45°)\text{A}$$

Q.2 (a) (i) Solution:

To obtain V_{TH}, we remove branch between ab and obtain V_a and V_b and the difference of the two voltages will give V_{TH}.

$$V_a = \frac{20}{20 + j20} \times 20 = 5 - j5$$

$$V_b = \frac{10}{-j20} \times -j10 = 5 + j0$$

$$V_{TH} = V_a - V_b = -j5 = 5\angle -90°$$

To obtain Z_{TH} we short circuit the voltage source and see the impedance between a and b.

$$Z_{TH} = Z_{ab} = Z_{ac} \parallel Z_{ad} + Z_{cb} \parallel Z_{bd}$$
$$= \frac{20 \times j20}{20 + j20} + \frac{(-j10)(-j10)}{-j20} = 10 + j10 - j5 = 10 + j5$$

Therefore,
$$I = \frac{V_{TH}}{Z_{TH} + 5} = \frac{5\angle -90°}{10 + j5 + 5}$$
$$= \frac{5\angle -90°}{15 + j5} = 0.316\angle -108° \text{A}$$

Q.2 (a) (ii) Solution:

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$175 \times 10^3 = \frac{1}{2\pi\sqrt{320\times10^{-12}L}}$$

$$L = 2.58 \text{ mH}$$

The reactance of the coil at resonance is

$$2\pi \times 175 \times 10^3 \times 2.58 \times 10^{-3} = 2840\ \Omega$$

Since $$Q = \frac{\omega_0 L}{R}$$

$$R = \frac{\omega_0 L}{Q} = \frac{2840}{50} = 56.8\ \Omega$$

The impedance of the circuit at resonance is

$$Z = R = 56.8\ \Omega$$

Therefore, $$\text{Current } I_0 = \frac{0.85}{56.8} = 14.96 \text{ mA}$$

The voltage across the capacitor = QV

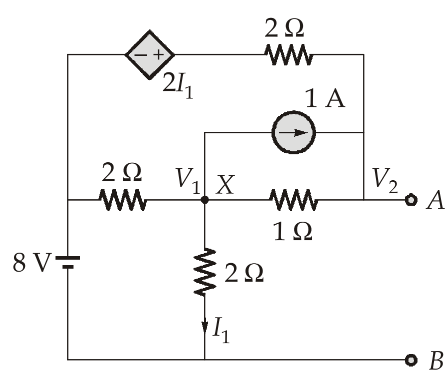
$$= 50 \times 0.85 = 42.5 \text{ V}$$

The bandwidth of the circuit is

$$\frac{f_0}{Q} = \frac{175\times10^3}{50} = 3.5 \text{ kHz}$$

Q.2 (b) Solution:

To calculate the Thevenin's voltage across A-B, the given circuit is redrawn after opening the resistance across the terminals A-B as shown in figure below,



Let the voltage at node X be V_1 and that at terminal A be V_2.

Using nodal analysis at node X, we obtain

$$\frac{8-V_1}{2} = \frac{V_1-V_2}{1}+\frac{V_1}{2}+1$$

$$2V_1 - V_2 = 3 \quad \text{...(i)}$$

Again using nodal analysis at terminal A, we obtain

$$\frac{8+2I_1-V_2}{2}+1+\frac{V_1-V_2}{1} = 0$$

$$3V_1 - 3V_2 = -10 \left(\because I_1 = \frac{V_1}{2}\right) \quad \text{...(ii)}$$

Solving the above equation (i) and (ii), we get

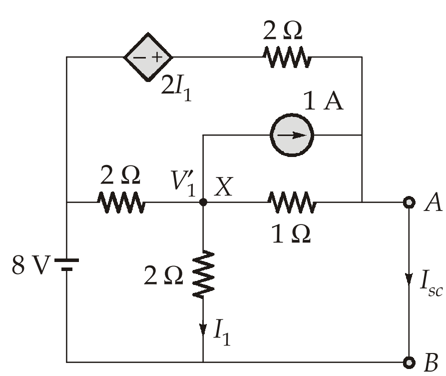
$$V_1 = \frac{19}{3}\text{ V} \quad \text{and} \quad V_2 = \frac{29}{3}\text{ V}$$

∴ The open circuit voltage,

$$V_{oc} = V_2 = \frac{29}{3}\text{ V} = 9.67\text{ V}$$

∴ Thevenin's voltage, $V_{th} = 9.67$ V

Let us now short circuit the terminals A-B, the circuit then becomes as shown in figure below,



Let the voltage at node X be V_1'. Using nodal analysis at node X, we obtain

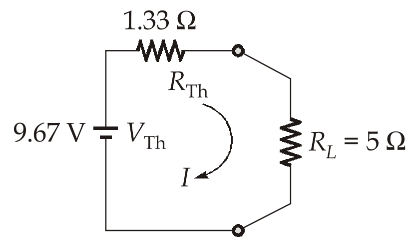
$$\frac{8-V_1'}{2} = \frac{V_1'}{2}+1+\frac{V_1'}{1}$$

$$V_1' = \frac{3}{2}\text{ V}$$

$$\text{Short circuit current, } I_{sc} = \frac{8+2I_1}{2}+1+\frac{V_1'-0}{1} = 7.25\text{ A} \quad \left[\because I_1 = \frac{V_1'}{2}\right]$$

$$\therefore \quad \text{Thevenin's resistance, } R_{th} = \frac{V_{oc}}{I_{sc}} = \frac{V_{Th}}{I_{sc}} = \frac{9.67}{7.25} = 1.33\ \Omega$$

The Thevenin equivalent of the given network is as shown below,



$$\therefore \quad \text{Current}, I = \frac{9.67}{1.33 + R_L} = \frac{9.67}{1.33+5} \approx 1.527 \text{ A}$$

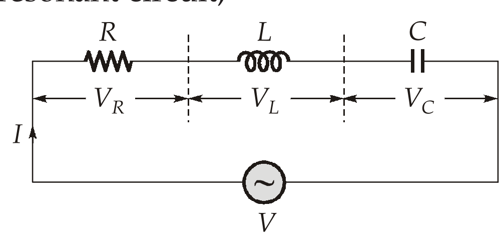
To get the maximum power transfer through R_L, as per maximum power transfer theorem,

$$R_L = R_{Th} = 1.33\ \Omega$$

$$\therefore \quad P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(9.67)^2}{4 \times 1.33} \approx 17.57 \text{ W}$$

Q.2 (c) Solution:

(i) Consider series RLC resonant circuit,



The voltage across inductor is V_L and is given by,

$$V_L = I \cdot (\omega L)$$

but

$$I = \frac{V}{Z}$$

$$v_L = \frac{V(\omega L)}{\left[R^2 + \left(\omega t - \frac{1}{\omega C}\right)^2\right]^{1/2}} \quad \text{...(i)}$$

For maximum voltage across inductor, differentiating equation (i), w.r.t. ω and equating it to zero

$$\frac{dV_L}{d\omega} = \frac{Lv}{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}} - \frac{1}{2}\frac{\omega L v\left[2\left(\omega L - \frac{1}{\omega C}\right)\left(L + \frac{1}{\omega^2 C}\right)\right]}{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{3/2}}$$

$$R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = \left(\omega L - \frac{1}{\omega C}\right)\left(L + \frac{1}{\omega^2 L}\right)\omega$$

$$\left(\omega L - \frac{1}{\omega C}\right)\left(\omega L + \frac{1}{\omega C}\right) - \left(\omega L - \frac{1}{\omega C}\right)^2 = R^2$$

$$(\omega L)^2 - \left(\frac{1}{\omega C}\right)^2 - (\omega L)^2 - \left(\frac{1}{\omega C}\right)^2 + \frac{2L}{C} = R^2$$

$$\frac{2}{(\omega C)^2} = \frac{2L}{C} - R^2$$

$$\frac{1}{(\omega C)^2} = \frac{L}{C} - \frac{R^2}{2}$$

$$\Rightarrow \quad \frac{1}{\omega^2} = LC - \frac{R^2 C^2}{2}$$

Put $\omega = \omega_{L\ max}$

$$\omega_{L\ max} = \frac{1}{\sqrt{LC - \frac{R^2C^2}{2}}} = \frac{\frac{1}{\sqrt{LC}}}{\sqrt{1 - \frac{R^2 C}{2L}}} = \frac{\omega_0}{\sqrt{1 - \frac{R^2 C}{2L}}}$$

Hence the frequency at which inductor voltage will be

$$\omega_{L\ max} = \frac{\omega_0}{\sqrt{1 - \frac{R^2 C}{2L}}}$$

(ii) Using the result of part-(i),

$$\omega_{L\ max} = \frac{1}{\sqrt{20 \times 10^{-6} \times 0.05}\left[1 - \frac{50^2 \times 20 \times 10^{-6}}{2 \times 0.05}\right]^{1/2}}$$

$$\omega_{L\ max} = 1414.21 \text{ rad/sec}$$

Voltage across inductor at $\omega = \omega_{L\ max}$ is given as

$$|V_L| = \frac{\omega_{L\max} \cdot L \cdot V}{\left[R^2 + \left(\omega_{L\max} L - \frac{1}{\omega_{L\max} C}\right)^2\right]^{1/2}}$$

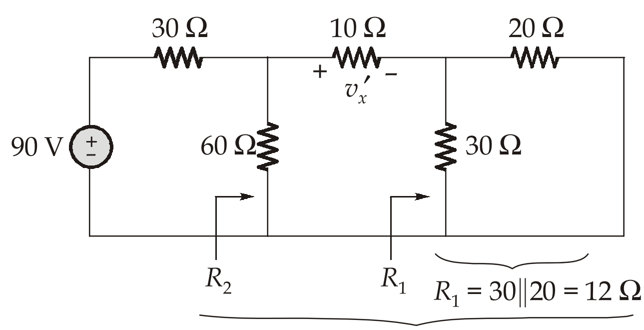
on putting the values,

$$|V_L|_{max} = \frac{1414.21 \times 0.05 \times 100}{\sqrt{50^2 + \left(1414.21 \times 0.05 - \frac{1}{1414.21 \times 20 \times 10^{-6}}\right)^2}}$$

$$|V_L|_{max} = 115.47 \text{ volts}$$

Q.3 (a) (i) Solution:

With 90 V source:



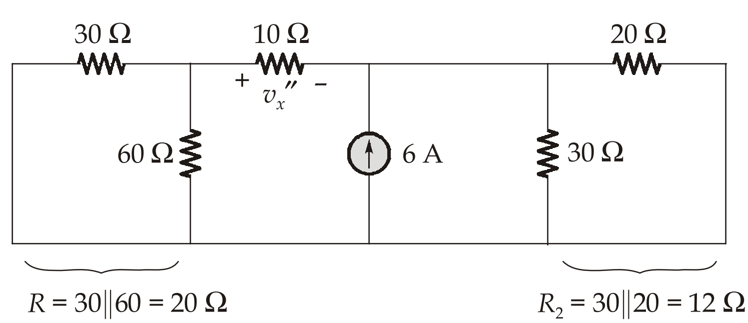
$$R_2 = 60 \,||\, (10 + R_1)$$
$$= 60 \,||\, (10 + 12)$$
$$= 16.098\ \Omega$$

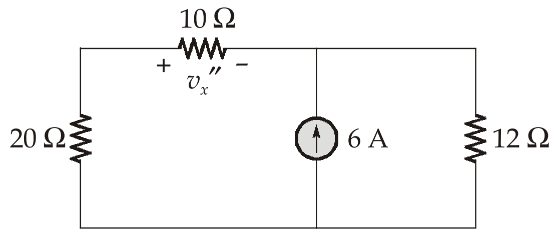
By voltage division,

$$V_{R_2} = \frac{R_2}{30 + R_2} \times 90 = \frac{16.098}{30 + 16.098} \times 90 = 31.429 \text{ Volts}$$

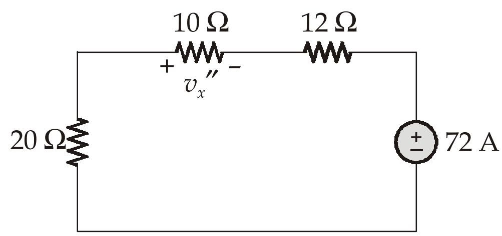
$$\therefore \quad v_x' = V_{R_2} \times \frac{10}{10 + R_1} = 31.429 \times \frac{10}{10 \times 12} = 14.286 \text{ Volts}$$

With 6 A source:



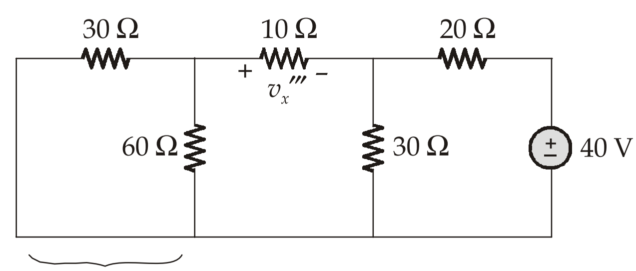


By source transformation

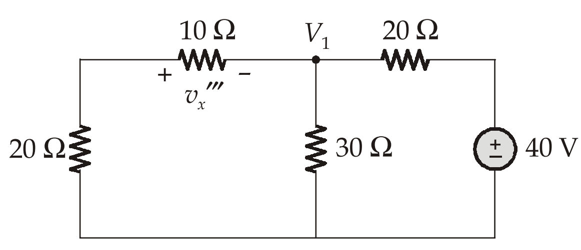


By Voltage division, $$v_x'' = (-72) \times \frac{10}{10+20+12} = -17.143 \text{ Volts}$$

With 40 V source :



$R = 30||60 = 20\ \Omega$



$R = (20+10)||30 = 15\ \Omega$

∴ $$V_1 = 40 \times \frac{15}{15+20} = 17.143 \text{ Volts}$$

So, $$V_x''' = (-V_1) \times \frac{10}{10+20} = -17.143 \times \frac{10}{10+20} = -5.714 \text{ Volts}$$

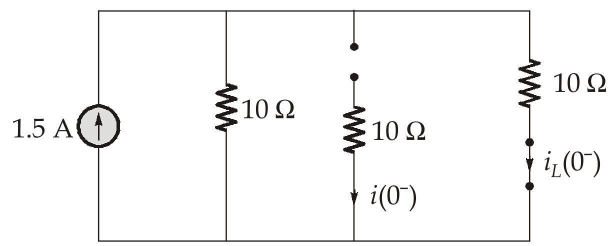
By Superposition Theorem,

$$v_x = v_x' + v_x'' + v_x'''$$

$$= 14.286 + (-17.143) + (-5.714) = -8.571 \text{ Volts}$$

Q.3 (a) (ii) Solution:

At $t = 0^-$, switch is open and the circuit in steady state hence, inductor acts as short-circuit.

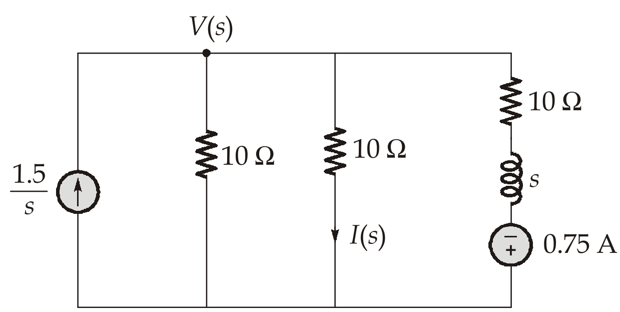


$$i_L(0^-) = \frac{1.5 \times 10}{10+10} = 0.75 \text{ A}$$

Since inductor current does not change instantly.

$$\therefore \quad i_L(0^-) = i_L(0^+) = 0.75 \text{ A}$$

For $t > 0$, the circuit is transformed into s-domain and is shown below,



$\therefore$ Applying KCL analysis at node $V(s)$,

$$\therefore \quad \frac{V(s)}{10} + \frac{V(s)}{10} + \frac{V(s)+0.75}{s+10} = \frac{1.5}{s}$$

$$V(s)\left[\frac{1}{10} + \frac{1}{10} + \frac{1}{s+10}\right] = \frac{1.5}{s} - \frac{0.75}{s+10}$$

$$V(s)\left[\frac{10+s+5}{5(s+10)}\right] = \frac{1.5(10+s) - 0.75s}{s(s+10)}$$

$$\therefore \quad V(s) = \frac{5(0.75s+15)}{s(s+15)} \qquad \text{...(i)}$$

Applying partial fraction expansion on equation (i) we get,

$$V(s) = \frac{A}{s} + \frac{B}{s+15}$$

$$A = \left.\frac{5(0.75s+15)}{s+15}\right|_{s=0} = 5$$

$$B = \left.\frac{5(0.75s+15)}{s}\right|_{s=-15} = -1.25$$

$$\therefore \quad V(s) = \frac{5}{s} - \frac{1.25}{s+15}$$

Now,

$$I(s) = \frac{V(s)}{10}$$

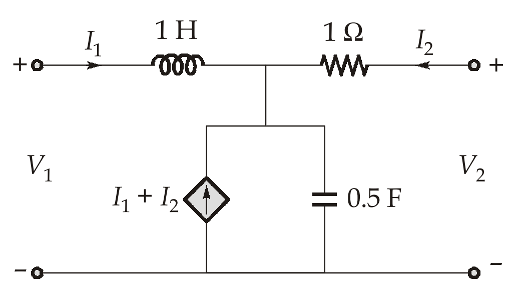
$$\therefore \quad I(s) = \frac{0.5}{s} - \frac{0.125}{s+15}$$

Now applying inverse Laplace transform, we get,

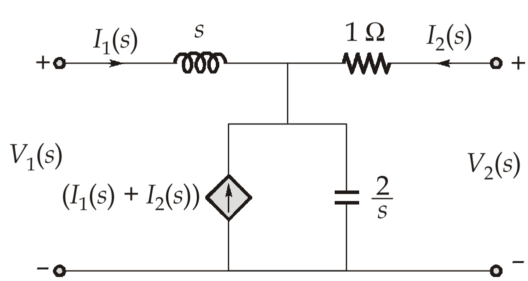
$$i(t) = (0.5 - 0.125e^{-15t}) \text{ A for } t > 0$$

Q.3 (b) Solution:

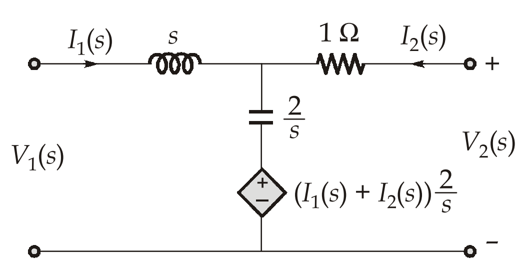
Given two port network,



By taking Laplace transform,



By using source transformation theorem, we can redraw the given two port network as



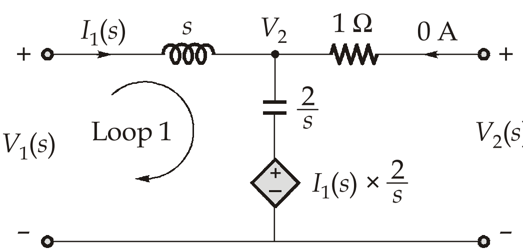
We know that,

transmission parameters can be written as,

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

For A:

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0}$$



By writing KVL in loop 1,

$$V_1(s) - I_1(s) \cdot s - V_2(s) = 0 \quad \text{...(i)}$$

where,
$$V_2(s) = \frac{2}{s} \cdot I_1(s) + \frac{2}{s} \cdot I_1(s)$$
$$V_2(s) = \frac{4}{s} I_1(s)$$
$$V_2(s) \cdot \frac{s}{4} = I_1(s) \quad \text{...(ii)}$$

Substituting equation (ii) in equation (i),

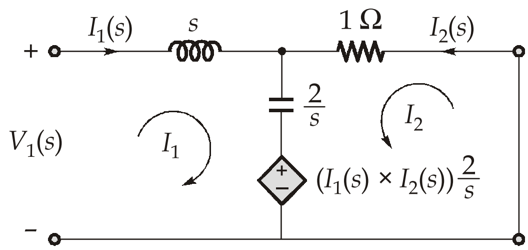
$$V_1(s) - \frac{s}{4} \cdot V_2(s) \cdot s - V_2(s) = 0$$
$$V_1(s) = \left(1 + \frac{s^2}{4}\right) V_2(s)$$

∴
$$A = \left.\frac{V_1(s)}{V_2(s)}\right|_{I_2=0} = 1 + \frac{s^2}{4}$$

For C:

From equation (ii),
$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{s}{4}$$

For B:
$$B = \left.\frac{-V_1}{I_2}\right|_{V_2=0}$$



By writing KVL in loop L_1:

$$V_1(s) - I_1(s)\cdot s - \frac{2}{s}(I_1(s)+I_2(s)) - (I_1(s)+I_2(s))\frac{2}{s} = 0$$

$$V_1(s) - I_1(s)\cdot s - \frac{4}{s}I_1(s) - \frac{4}{s}I_2(s) = 0$$

$$V_1(s) - \left(s+\frac{4}{s}\right)I_1(s) - \frac{4}{s}I_2(s) = 0 \quad \text{...(iii)}$$

by writing KVL in loop L_2:

$$0 - I_2(s)\cdot 1 - (I_1(s)+I_2(s))\frac{2}{s} - (I_1(s)+I_2(s))\frac{2}{s} = 0$$

$$-I_1(s)\left(\frac{4}{s}\right) - I_2(s)\left[1+\frac{4}{s}\right] = 0$$

$$\frac{4}{s}\cdot I_1(s) = -I_2(s)\left[\frac{s+4}{s}\right]$$

$$I_1(s) = -I_2(s)\left[\frac{s+4}{4}\right] \quad \text{...(iv)}$$

Substituting equation (iv) in equation (iii),

$$V_1(s) + \left(\frac{s^2+4}{s}\right)\left(\frac{s+4}{4}\right)I_2(s) - \frac{4}{s}I_2(s) = 0$$

$$V_1(s) + \left(\frac{s^3+4s^2+4s+16}{4s} - \frac{4}{s}\right)I_2(s) = 0$$

$$V_1(s) + \frac{s^3+4s^2+4s+16-16}{4s}I_2(s) = 0$$

$$V_1(s) = -\left(\frac{s^3+4s^2+4s}{4s}\right)I_2(s) = -\left(\frac{s^2+4s+4}{4}\right)I_2(s)$$

$\therefore$
$$B = \left.\frac{-V_1}{I_2}\right|_{V_2=0} = \left(\frac{s^2+4s+4}{4}\right)$$

For D:

From equation (iv)
$$D = \left.\frac{-I_1}{I_2}\right|_{V_2=0} = \left(\frac{s+4}{4}\right)$$

$\therefore$ The transmission parameters matrix is

$$[T] = \begin{bmatrix} 1+\frac{s^2}{4} & \left(\frac{s^2+4s+4}{4}\right) \\ \frac{s}{4} & \left(\frac{s+4}{4}\right) \end{bmatrix}$$

Q.3 (c) (i) Solution:

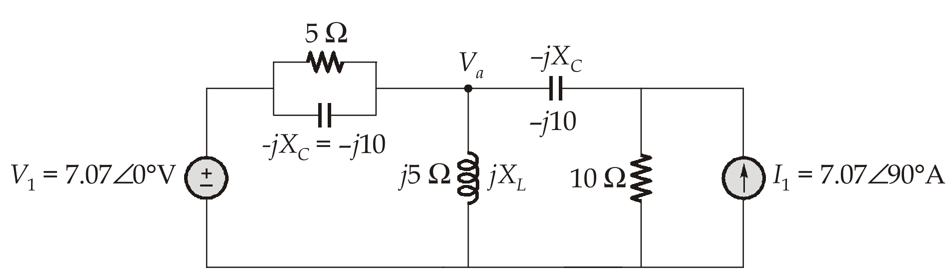
$$V_1 = \frac{10}{\sqrt{2}}\angle 0° V = 7.07\angle 0° V$$

$\therefore$
$$I_1 = \frac{10}{\sqrt{2}}\angle 90° A = 7.07\angle 90° A$$

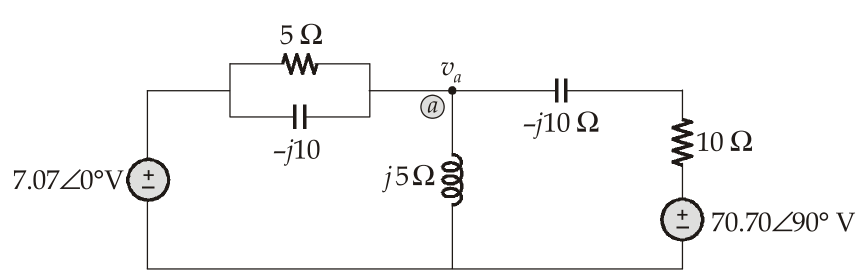
$$X_C = \frac{1}{\omega C} = \frac{1}{10^6 \times 0.1 \times 10^{-6}} = 10\,\Omega$$

$$X_L = 10^6 \times 5 \times 10^{-6} = 5\,\Omega$$

Phasor domain circuit will be



By source transformation



By KCL at node - a,

$$\frac{V_a - 7.07\angle 0°}{5} + \frac{V_a - 7.07\angle 0°}{(-j10)} + \frac{V_a}{(j5)} + \frac{V_a - 70.7\angle 90°}{(10 - j10)} = 0$$

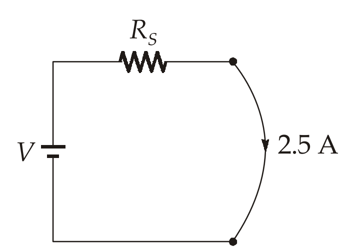
$$V_a\left[\frac{1}{5} + \frac{1}{-j10} + \frac{1}{j5} + \frac{1}{10 - j10}\right] = \frac{7.07}{5}\angle 0° + \frac{7.07\angle 0°}{(-j10)} + \frac{70.7\angle 90°}{(10 - j10)}$$

$$\Rightarrow \qquad V_a = 18.602\angle 127.87° \text{ Volts}$$

$$\therefore \qquad V_a(t) = \sqrt{2} \times 18.602 \sin\left(10^6 t + 127.87°\right) \text{V}$$

$$= 26.307 \sin\left(10^6 t + 127.87°\right) \text{V}$$

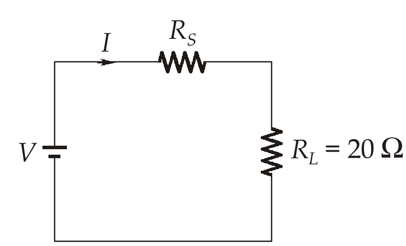
Q.3 (c) (ii) Solution:



Momentary short circuit

$$\frac{V}{R_S} = 2.5 \text{ A} \qquad \text{...(i)}$$

Figure shows a practical voltage source of V volts possess an internal series resistance of R_S Ω provides a current of 2.5 A when (momentarily) short circuited,



$$\text{Power consumed} = 80 \text{ W}$$

$$I^2 R_L = \left(\frac{V}{R_S + R_L}\right)^2 \times R_L = \frac{V^2}{(R_S + R_L)^2} \times R_L = 80$$

$$\frac{V^2 \times 20}{(R_S + 20)^2} = 80$$

$$V^2 = \frac{80}{20}(R_S + 20)^2$$

$$= R_S^2(2.5)^2 \qquad ...(ii)$$

$$4(R_S + 20)^2 = 6.25\, R_S^2$$

$$4(R_S^2 + 20^2 + 40R_S) = 6.25\, R_S^2$$

$$2.25R_S^2 - 160R_S - 1600 = 0 \qquad ...(iii)$$

By solving the quadratic equation, we get

$$R_S = 80\ \Omega$$

1. The open circuit - voltage,

$$V = 2.5 \times 80 = 200\ \text{V}$$

2. The maximum power delivered to this R_L

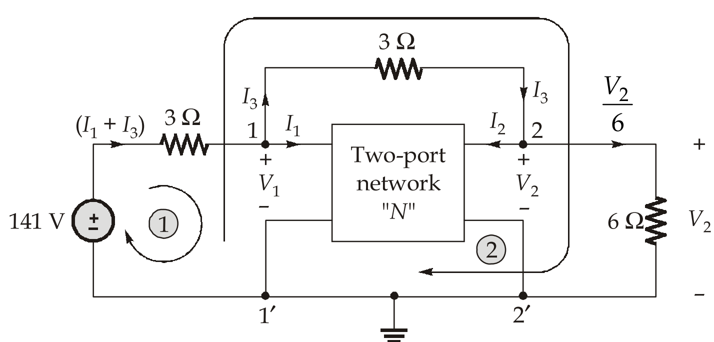
$$= \left(\frac{V}{R_S + R_L}\right)^2 \times R_L = \frac{V^2}{4R_L} = 125\ \text{W}$$

3. The value of R_L to which it can deliver a maximum power is equal to that of R_S

$$R_L = R_S = 80\ \Omega$$

Q.4 (a) (i) Solution:

- Considering the given network as shown below:



- From the given z-parameters of the two-port network "N", we can write,

$$V_1 = 2I_1 + I_2 \qquad ...(i)$$

$$V_2 = I_1 + 4I_2 \qquad ...(ii)$$

- By applying KVL in loop (1), we get,

$$3(I_1 + I_3) + V_1 = 141\ \text{V}$$

$$3I_1 + 3I_3 + 2I_1 + I_2 = 141$$

$$5I_1 + I_2 + 3I_3 = 141 \qquad ...(iii)$$

- By applying KVL in loop (2), we get,

$$V_1 - V_2 = (3)I_3$$
$$(2I_1 + I_2) - (I_1 + 4I_2) = 3I_3$$
$$I_1 - 3I_2 - 3I_3 = 0 \quad \text{...(iv)}$$

- By applying KCL at node (2), we get,

$$I_3 = I_2 + \frac{V_2}{6}$$
$$6I_3 = 6I_2 + (I_1 + 4I_2)$$
$$I_1 + 10I_2 - 6I_3 = 0 \quad \text{...(v)}$$

- By solving the equations (iii), (iv) and (v), we get,

$$I_1 = 24 \text{ A}$$
$$I_2 = 1.5 \text{ A}$$
$$I_3 = 6.5 \text{ A}$$

Q.4 (a) (ii) Solution:

$$V_R(0^+) = 10 \text{ V}$$
$$i_L(0^+) = \frac{10}{R_2} \quad \text{...(i)}$$

As the switch is closed at $t = 0$ sec, i_L by passes R_1

Hence, the decaying i_L is through R_2 only

$$\tau = \frac{L}{R_2} = \frac{20\,\text{m}}{R_2} \text{ for } t > 0 \quad \text{...(ii)}$$
$$V_R(1 \text{ ms}) = 5 \text{ V} \quad \text{...(iii)}$$
$$V_R(1 \text{ ms}) = i_L(1 \text{ ms}) \times R_2$$
$$= R_2 \times i_L(0^+)\, e^{\frac{-1\times 10^{-3} \times R_2}{20 \times 10^{-3}}}$$
$$= R_2 \times \frac{10}{R_2} e^{\frac{-R_2}{20}} \quad \text{...(iv)}$$

By equating equation (iii) and (iv), we get

$$10 e^{\frac{-R_2}{20}} = 5$$
$$e^{\frac{-R_2}{20}} = \frac{5}{10}$$

$$\frac{-R_2}{20} = \ln(0.5) = -0.693$$

$$R_2 = 13.86\ \Omega$$

At $t = 0^-$,

$$\text{Voltage across } R_1 = V_R = 10\text{ V} \quad \text{...(vi)}$$

∵ Inductor acts as a short circuit,

$$\text{Current through } R_1 = 2 - \frac{V_R}{R_2} = 2 - \frac{10}{13.86} = 1.28\text{ A} \quad \text{...(vii)}$$

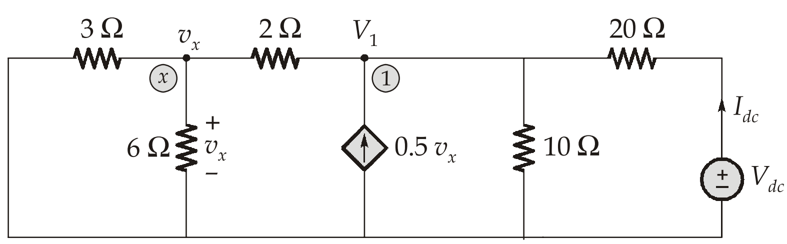
By substituting equation (vii) in equation (vi)

$$1.28R_1 = 10$$

$$R_1 = 7.82\ \Omega$$

Q.4 (b) Solution:

To determine R_{Th}, deactivate all the independent voltage source by short circuit and independent current source by open circuit & connect a test voltage source V_s delivering current I_S at terminals a - b.



By KCL at node (x)

$$\frac{v_x}{3} + \frac{v_x}{6} + \frac{v_x - v_1}{2} = 0$$

$$v_x\left[\frac{1}{3} + \frac{1}{6} + \frac{1}{2}\right] = \frac{v_1}{2}$$

$$v_x = 0.5v_1 \quad \text{...(i)}$$

KCL at node-1

$$\frac{v_1 - v_x}{2} + \frac{v_1}{10} = I_{dc} ++ 0.5v_x$$

$$0.6v_1 = I_{dc} + V_x \quad \text{...(ii)}$$

From equation (ii),

$$0.1v_1 = I_{dc} \quad \text{...(iii)}$$

Now,

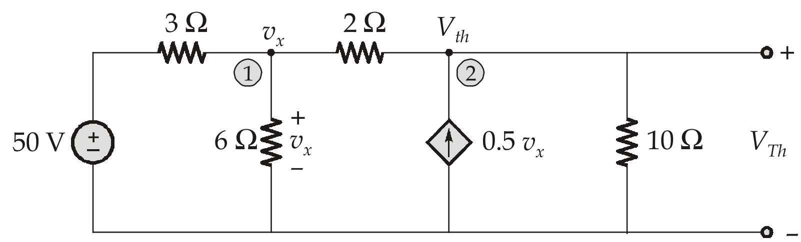
$$\frac{V_{dc} - v_1}{20} = I_{dc}$$

$$\frac{V_{dc} - 10I_{dc}}{20} = I_{dc}$$

$$V_{dc} = 30I_{dc}$$

$$R_{Th} = \frac{V_{dc}}{I_{dc}} = 30\ \Omega$$

Calculation of V_{Th} :



By KCL at node (1)

$$\frac{v_x - 50}{3} + \frac{v_x}{6} + \frac{v_x - V_{Th}}{2} = 0$$

$$v_x\left[\frac{1}{3} + \frac{1}{6} + \frac{1}{2}\right] = \frac{V_{Th}}{2} + \frac{50}{3}$$

$$\Rightarrow \quad v_x = 0.5\ V_{Th} + 16.667 \quad \ldots(i)$$

By KCL at node (2)

$$\frac{V_{Th} - v_x}{2} + \frac{V_{Th}}{10} - 0.5v_x = 0$$

$$V_{Th}\left[\frac{1}{2} + \frac{1}{10}\right] = v_x \quad \ldots(ii)$$

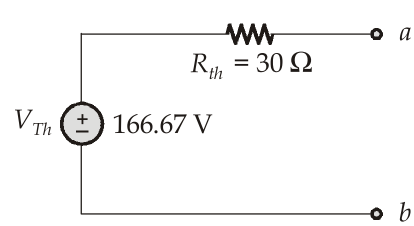
From (i) and (ii)

$$0.5\ V_{Th} + 16.667 = 0.6\ V_{Th}$$

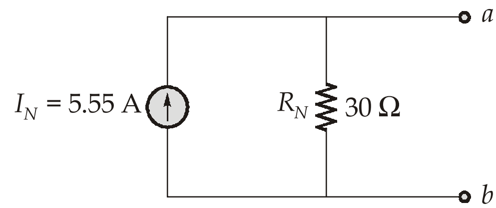
$$\Rightarrow \quad V_{Th} = 166.67 \text{ volts}$$

Thus, Norton's current, $I_N = \dfrac{V_{Th}}{R_{Th}} = \dfrac{166.67}{30} = 5.55$ A

Thevenin's equivalent circuit is

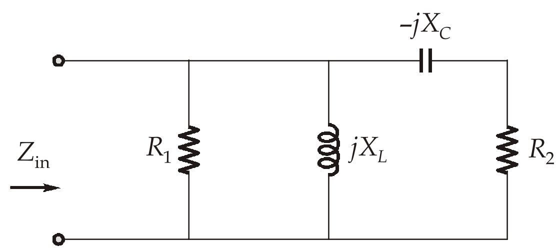


Norton's equivalent circuit is



Q.4 (c) (i) Solution:

At frequency, ω rad/s, the reactances are:



Where, $R_1 = 1\ \Omega,\ R_2 = 0.1\ \Omega,\ X_L = \omega_0 L\ ;\ X_C = \dfrac{1}{\omega_0 C}$

$$Y_{in} = \frac{1}{R_1} + \frac{1}{jX_L} + \frac{1}{R_2 - jX_C}$$

$$= \frac{1}{R_1} + \frac{1}{jX_L} + \frac{R_2 + jX_C}{R_2^2 - X_C^2}$$

For the circuit to resonate,

$$\text{Img}\,[Y_{in}] = 0$$

$$\text{Img}\,[Y_{in}] = \frac{1}{jX_L} + \frac{jX_C}{R_2^2 + X_C^2} = 0$$

$$\frac{1}{X_L} = \frac{X_C}{R_2^2 + X_C^2}$$

$\Rightarrow$
$$R_2^2 + X_C^2 = X_L X_C$$

$$R_2^2 = X_L X_C - X_C{}^2$$

$$R_2^2 = \frac{L}{C} - \frac{1}{(\omega_0 C)^2}$$

$\Rightarrow$
$$\omega_0 = \frac{1}{\sqrt{LC - R_2^2 C^2}}$$

$$\Rightarrow \qquad \omega_0 = \frac{1}{\sqrt{20\times10^{-3}\times9\times10^{-6} - (0.1\times9\times10^{-6})^2}}$$

$$\Rightarrow \qquad \omega_0 = 2357.02 \text{ rad/sec}$$

Therefore resonance frequency is,

$$\omega_0 = 2357.02 \text{ rad/sec}$$

$\Rightarrow$ At $\omega = \omega_0$,

$$Y_{in} = \frac{1}{R_1} + \frac{R_2}{R_2^2 + (X_C)^2}$$

$$= \frac{1}{1} + \frac{0.1}{(0.1)^2 + (2357.02\times10^{-6})^2} = 10.56 \text{ Mho}$$

Therefore input impedance at $\omega = \omega_0$, is

$$Z_{in} = \frac{1}{Y_{in}} = \frac{1}{10.56} = 0.09461\ \Omega$$

Q.4 (c) (ii) Solution:

For the given graph, Kirchhoff's current law for the branch currents ($i_1, i_2, \ldots\ldots i_6$) gives the equations

$$i_1 + i_2 + i_6 = 0\,; \qquad -i_1 + i_3 - i_5 = 0$$

$$-i_2 - i_3 + i_4 = 0\,; \qquad -i_4 + i_5 - i_6 = 0$$

In matrix form, these equation can be represented as

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = 0$$

The complete incidence matrix is

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix}$$

So, the reduced incidence matrix is

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \end{bmatrix}$$

Thus the number of possible trees of the graph

$$= \det\left\{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right\}$$

$$= \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 16$$

Section B : Engineering Mathematics

Q.5 (a) Solution:

We have, $$\frac{dy}{dx} = \frac{y+x+2}{y-x+1}$$

Put $x = X + h$, $y = Y + k$

The given equation reduces to

$$\frac{dY}{dX} = \frac{(Y+k)+(X+h)+2}{(Y+k)-(X+h)+1} = \frac{X+Y+(h+k+2)}{Y-X+(k-h+1)}$$

Now, choose h and k so that $h + k + 2 = 0$, $k - h + 1 = 0$.

Solving these equations, we get

$$h = -\frac{1}{2},\ k = -\frac{3}{2}$$

$$\therefore \quad \frac{dY}{dX} = \frac{X+Y}{Y-X}$$

Put $Y = VX$ so that $\frac{dY}{dX} = V + X\frac{dV}{dX}$ ...(ii)

The equation (ii) is transformed as

$$V + X\frac{dV}{dX} = \frac{X+VX}{VX-X} = \frac{1+V}{V-1}$$

$$X\frac{dV}{dX} = \frac{1+V}{V-1} - V = \frac{-V^2+V+1+V}{V-1} = \frac{-V^2+2V+1}{V-1}$$

$$\frac{1}{2}\int\left(\frac{2V-2}{V^2-2V-1}\right)dV = -\int\frac{dX}{X}$$

$$\frac{1}{2}\log(V^2-2V-1) = -\log X + \log C$$

$$(V^2-2V-1)^{\frac{1}{2}} = -\log X + \log C$$

$$(V^2-2V-1)^{\frac{1}{2}} = \frac{C}{X}$$

$$X^2(V^2-2V-1) = C_1$$

$$\left(y+\frac{3}{2}\right)^2 - 2\left(x+\frac{1}{2}\right)\left(y+\frac{3}{2}\right) - \left(x+\frac{1}{2}\right)^2 = C_1$$

Q.5 (b) Solution:

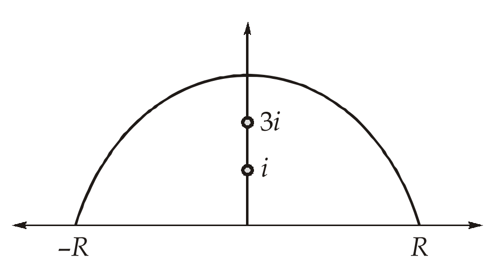
Consider $\int_C \frac{z^2-z+2}{z^4+10z^2+9}dz = \int_C f(z)dz$ where C is the contour consisting of the upper half of a large circle $|z| = R$ and the real axis from $-R$ to R.

By Residue Theorem,

$$\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{CR} f(z)\,dz$$

$$= 2\pi i \text{ (sum of residues)} \qquad \text{...(i)}$$

Poles of $f(z)$ are given by $z^4 + 10z^2 + 9 = 0$



$$(z^2+1)(z^2+9) = 0$$
$$z = \pm i \text{ and } z = \pm 3i$$

Thus, $z \;\; \pm i$ and $z = \pm 3i$

are the simple poles of which only $z = i$ and $z = 3i$ lie inside C.

Sum of residue at these poles.

$$= \lim_{z\to i}(z-i)f(z) + \lim_{z\to 3i}(z-3i)f(z)$$

$$= \lim_{z\to i}\frac{(z-i)(z^2-z+2)}{(z-i)(z+i)(z^2+9)} + \lim_{z\to 3i}\frac{(z-3i)(z^2-z+2)}{(z-3i)(z^2+1)(z+3i)}$$

$$= \frac{i^2-i+2}{(i^2+9)(2i)} + \frac{9i^2-3i+2}{(9i^2+1)6i}$$

$$= \frac{1-i}{16i} + \frac{7+3i}{48i} = \frac{10}{48i} = \frac{5}{24i}$$

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \times \frac{5}{24i} = \frac{5\pi}{12}$$

Q.5 (c) Solution:

(i)

$$\text{div } \overline{F} = \nabla\cdot\overline{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2\hat{i} + z\hat{j} + yz\hat{k})$$
$$= 2x + y$$

By Divergence Gauss Theorem

$$\iint F\cdot\hat{n}\,ds = \iiint \text{div } F = \iiint (2x+y)dx\,dy\,dz$$

$$= \iint dx\,dy\int_0^c (2x+y)dz = \iint dx\,dy\,[2xz+yz]_0^c$$

$$= \iint dx\,dy\,[2cx+cy]$$

$$= \int dx\left[2cxy + c\frac{y^2}{2}\right]_0^b$$

$$= \int_0^a \left[2cbx + \frac{cb^2}{2}x\right]_0^a = \frac{abc}{2}(2a+b)$$

Q.5 (c) (ii) Solution:

1. Given : $f(x) = 2x^3 - 9x^2 + 12x - 5$ ...(i)

It is differentiable for all x in [0, 3]. Since, it is a polynomial differentiating (i) w.r.t. x, we get

$$f'(x) = 2 \times 3x^2 - 9 \times 2x + 12 = 6(x^2 - 3x + 2)$$

Now, $f'(x) = 0$

$$6(x^2 - 3x + 2) = 0$$

$$(x - 1)(x - 2) = 0$$

$$x = 1, 2$$

Also, 1, 2 both are in [0, 3]. Therefore, 1 and 2 both are stationary points.

Further, $f(1) = 2 \times 1^3 - 9 \times 1^2 + 12 \times 1 - 5 = 0$

$$f(2) = 2 \times 2^3 - 9 \times 2^2 + 12 \times 2 - 5$$

$$= 16 - 36 + 24 - 5 = -1$$

Also, $f(0) = -5$

and $f(3) = 2.3^3 - 9.3^2 + 12.3 - 5$

$$= 54 - 81 + 36 - 5 = 4$$

Therefore, the absolute maxima value is 4 and the absolute minima value is –5. The point of maxima is 3 and the point of minima is 0.

2. Given : $f(x) = 12x^{4/3} - 6x^{1/3}, x \in [-1, 1]$

Differentiating (i) w.r.t. x, we get

$$f'(x) = 12 \cdot \frac{4}{3} x^{1/3} - 6 \cdot \frac{1}{3} x^{-2/3}$$

$$= 16x^{1/3} - \frac{2}{x^{2/3}} = \frac{2(8x-1)}{x^{2/3}}$$

Now, $f'(x) = 0$

$$\frac{2(8x-1)}{x^{2/3}} = 0$$

$$x = \frac{1}{8}$$

As $\frac{1}{8} \in [-1, 1]$, $\frac{1}{8}$ is a critical point.

Also, we note that f is not differentiable at $x = 0$.

$$f\left(\frac{1}{8}\right) = 12\left(\frac{1}{8}\right)^{4/3} - 6\left(\frac{1}{8}\right)^{1/3}$$

$$= 12\left(\frac{1}{2}\right)^{4} - 6\cdot\frac{1}{2} = 12\cdot\frac{1}{16} - 3 = \frac{-9}{4}$$

$$f(0) = 12.0 - 6.0 = 0$$

$$f(-1) = 12(-1)^{4/3} - 6(-1)^{1/3}$$

$$= 12.1 - 6(-1)$$

$$= 18$$

$$f(1) = 12.1^{4/3} - 6.1^{1/3}$$

$$= 12 - 6$$

$$= 6$$

Therefore, the absolute maxima value is 18 and the absolute minima value is $-\frac{9}{4}$.

The point of maxima is –1 and the point of minima is $\frac{1}{8}$.

Q.5 (d) Solution:

It is clear that the given function

$$f(x) = \begin{cases} (1-x) & , \quad x<1 \\ (1-x)(2-x), & 1 \le x \le 2 \\ (3-x) & , \quad x>2 \end{cases}$$

Continuous and differentiable at all points except possibly at $x = 1$ and 2.

Continuity at $x = 1$

$$\text{LHL} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(1-x)$$

$$= \lim_{h\to 0}[1-(1-h)] = \lim_{h\to 0} h = 0$$

and

$$\text{RHL} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(1-x)(2-x)$$

$$= \lim_{h\to 0}[1-(1+h)][2-(1+h)]$$

$$= \lim_{h\to 0} -h\cdot(1-h) = 0$$

$\therefore$ LHL = RHL = $f(1) = 0$

Therefore, f is continuous at $x = 1$,

$$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{1-(1-h)-0}{-h} = \lim_{h \to 0}\left(\frac{h}{-h}\right) = 1$$

and

$$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{[1-(1+h)][2-(1+h)]-0}{h}$$

$$= \lim_{h \to 0} \frac{-h(1-h)}{h} = \lim_{h \to 0}(h-1) = -1$$

Since, $L[f'(1)] = Rf'(1)$, therefore f is differentiable.

at $x = 1$

Continuity at $x = 2$,

$$\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (1-x)(2-x)$$

$$= \lim_{h \to 0} [-1-(2-h)][2-(2-h)]$$

$$= \lim_{h \to 0} (-1+h)h = 0$$

and

$$\text{RHL} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3-x)$$

$$= \lim_{h \to 0} [3-(2+h)]$$

$$= \lim_{h \to 0} (1-h) = 1$$

Since, LHL $\neq$ RHL, therefore f is not continuous at $x = 2$ as such f cannot be differentiable at $x = 2$.

Hence, f is continuous and differentiable at all points except at $x = 2$.

Q.5 (e) Solution:

Since the total probability is unity.

$$\therefore \quad \int_0^6 f(x)\,dx = 1$$

$$\int_0^2 kx\,dx + \int_2^4 2k\,dx + \int_4^6 (-kx+6k)\,dx = 1$$

$$k\left|\frac{x^2}{2}\right|_0^2 + 2k|x|_2^4 + \left(-\frac{kx^2}{2}+6kx\right)_4^6 = 1$$

$$2k + 4k + (-10k + 12k) = 1$$

i.e., $$k = \frac{1}{8}$$

$$\text{Mean of } X = \int_0^6 xf(x)\,dx$$

$$= \int_0^2 kx^2\,dx + \int_2^4 2kx\,dx + \int_4^6 x(-kx+6k)\,dx$$

$$= k\left|\frac{x^3}{3}\right|_0^2 + 2k\left|\frac{x^2}{2}\right|_2^4 + (-k)\left|\frac{x^3}{3}\right|_4^6 + 6k\left|\frac{x^2}{2}\right|_4^6$$

$$= k\left(\frac{8}{3}\right) + k(12) - k\left(\frac{152}{3}\right) + 3k(20) = \frac{24}{8} = 3$$

Q.6 (a) (i) Solution:

According to this theorem,

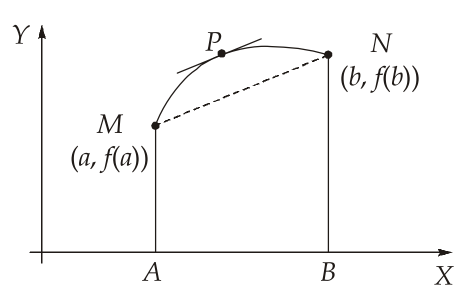
If a function $f(x)$ is such that

(a) It is continuous in closed interval $[a, b]$

(b) It is derivable in open interval (a, b)

then there exist at least one value 'C' of x lying in (a, b) such that,

$$\frac{f(b)-f(a)}{b-a} = f'(C)$$



Geometrical significance: If the graph of a function is a continuous curve from M to N and has a unique tangent at every point between M and N, then there exists at least one point 'P' in the curve such that the tangent at P is parallel to the chord MN joining its extremities.

Q.6 (a) (ii) Solution:

Given equation in symbolic form is,

$$(D^2 - 2D + 2)y = x + e^x \cos x$$

To find complementary function:

Its auxiliary equation is

$$D^2 - 2D + 2 = 0$$

$$\therefore \quad D = \frac{2 \pm \sqrt{(4-8)}}{2} = 1 \pm i$$

Thus complementary function = $e^x(c_1 \cos x + c_2 \sin x)$

To find particular integral,

$$\text{P.I.} = \frac{1}{D^2 - 2D + 2}(x) + \frac{1}{D^2 - 2D + 2}(e^x \cos x)$$

$$= \frac{1}{2}\left[1 - \left(D - \frac{D^2}{2}\right)\right]^{-1}(x) + e^x \frac{\cos x}{(D+1)^2 - 2(D+1) + 2}$$

$$= \frac{1}{2}\left(1 + D - \frac{D^2}{2}\right)x + e^x \frac{1}{D^2 + 1}\cos x$$

$$= \frac{1}{2}(x + 1 - 0) + e^x \cdot x \frac{1}{2D}\cos x$$

$$= \frac{(x+1)}{2} + \frac{xe^x}{2}\int \cos x \, dx = \frac{1}{2}(x+1) + \frac{xe^x}{2}\sin x$$

Hence the complete solution is,

$$y = e^x(c_1 \cos x + c_2 \sin x) + \frac{1}{2}(x+1) + \frac{xe^x}{2}\sin x$$

Q.6 (a) (iii) Solution:

Given matrix, $$A = \begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{bmatrix}$$

Conjugate of A, i.e., $$\bar{A} = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(-1-i) \\ \frac{1}{2}(1-i) & \frac{1}{2}(1+i) \end{bmatrix}$$

∴ Transpose of $\bar{A}$ i.e.,

$$(\bar{A})^T = A^\theta = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1-i) & \frac{1}{2}(1+i) \end{bmatrix} \quad ...(i)$$

Now,

$$A^\theta \cdot A = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1-i) & \frac{1}{2}(1+i) \end{bmatrix}\begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}(1+1)+\frac{1}{4}(1+1) & -\frac{1}{4}(1-i)^2+\frac{1}{4}(1-i)^2 \\ -\frac{1}{4}(1+i)^2+\frac{1}{4}(1+i)^2 & \frac{1}{4}(1+1)+\frac{1}{4}(1+1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarly, $AA^\theta = I$

Hence, A is a unitary matrix.

Also,

$$A^{-1} = A^\theta = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ -\frac{1}{2}(1+i) & \frac{1}{2}(1+i) \end{bmatrix}$$

Q.6 (b) (i) Solution:

Here, we have

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$5I = A^2 - 4A$$

Multiplying by A^{-1} we get

$$5A^{-1} = A - 4I$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

Q.6 (b) (ii) Solution:

Consider the matrix equation,

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4 = 0$$

$$\lambda_1(1, 2, 4) + \lambda_2(2, -1, 3) + \lambda_3(0, 1, 2) + \lambda_4(-3, 7, 2) = 0$$

$$\lambda_1 + 2\lambda_2 + 0\lambda_3 - 3\lambda_4 = 0$$

$$2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 = 0$$

$$4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$$

This is the homogeneous system,

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ and $R_1 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0$$

$$-5\lambda_2 + \lambda_3 + 13\lambda_4 = 0$$

$$\lambda_3 + \lambda_4 = 0$$

Let, $\lambda_4 = t,\ \lambda_3 + t = 0,\ \lambda_3 = -t,$

$$-5\lambda_2 - t + 13t = 0$$

$$\lambda_2 = \frac{12}{5}t$$

$$\lambda_1 + \frac{24t}{5} - 3t = 0$$

$$\lambda_1 = \frac{-9t}{5}$$

Hence, the given vectors are linearly dependent, substituting the values of λ in (i),

We get,

$$\frac{-9t}{5}X_1 + \frac{12t}{5}X_2 - tX_3 + tX_4 = 0$$

$$9X_1 - 12X_2 + 5X_3 - 5X_4 = 0$$

Q.6 (c) (i) Solution:

To find complementary function:

Its auxiliary equation is,

$$D^2 - 4D + 4 = 0$$

i.e., $$(D-2)^2 = 0$$

$\therefore$ $$D = 2, 2$$

$\therefore$ Complementary function = $(C_1 + C_2x)\,e^{2x}$

To find particular integral,

$$\text{P.I.} = \frac{1}{(D-2)^2}(8x^2e^{2x}\sin 2x)$$

$$= 8e^{2x}\frac{1}{(D+2-2)^2}(x^2\sin 2x)$$

$$= 8e^{2x}\frac{1}{D^2}(x^2\sin 2x)$$

$$= 8e^{2x}\frac{1}{D}\int x^2\sin 2x\,dx$$

$$= 8e^{2x}\frac{1}{D}\left\{x^2\left(\frac{-\cos 2x}{2}\right) - \int 2x\left(\frac{-\cos 2x}{2}\right)dx\right\}$$

$$= 8e^{2x}\frac{1}{D}\left\{-\frac{x^2}{2}\cos 2x+\frac{x\sin 2x}{2}-\int 1\cdot\frac{\sin 2x}{2}\,dx\right\}$$

$$= 8e^{2x}\int\left\{-\frac{x^2}{x}\cos 2x+\frac{x}{2}\sin 2x+\frac{\cos 2x}{4}\right\}dx$$

$$= 8e^{2x}\left[\left\{-\frac{x^2}{2}\frac{\sin 2x}{2}-\int(-x)\frac{\sin 2x}{2}\,dx\right\}+\left\{\int\frac{x}{2}\sin 2x\,dx\right\}+\frac{\sin 2x}{8}\right]$$

$$= 8e^{2x}\left[\left(-\frac{x^2}{4}+\frac{1}{8}\right)\sin 2x+\int x\sin 2x\,dx\right]$$

$$= 8e^{2x}\left[\left(\frac{1}{8}-\frac{x^2}{4}\right)\sin 2x+x\left(-\frac{\cos 2x}{2}\right)-\int 1\cdot\left(-\frac{\cos 2x}{2}\right)dx\right]$$

$$= 8e^{2x}\left[\left(\frac{1}{8}-\frac{x^2}{4}\right)\sin 2x-\frac{x\cos 2x}{2}+\frac{\sin 2x}{4}\right]$$

$$= e^{2x}[(3-2x^2)\sin 2x-4x\cos 2x]$$

∴ The complete solution is,

$$y = e^{2x}[C_1 + C_2x + (3 - 2x^2)\sin 2x - 4x(\cos 2x)]$$

Q.6 (c) (ii) Solution:

S. no.	x	y	xy	x^2
1	1	1	1	1
2	3	2	6	9
3	4	4	16	16
4	6	4	24	36
5	8	5	40	64
6	9	7	63	81
7	11	8	88	121
8	14	9	126	196
Total	56	40	364	524

Let $y = a + bx$ be the line of regression of y on x, where a and b are given by the following equations,

$$\Sigma y = na + b\Sigma x$$

or $$40 = 8a + 56b \quad \text{...(i)}$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

or $$364 = 56a + 524b \quad \text{...(ii)}$$

On solving (i) and (ii), we get

$$a = \frac{6}{11}$$

and $$b = \frac{7}{11}$$

The equation of the required line is

$$y = \frac{6}{11} + \frac{7}{11}x$$

or $$7x - 11y + 6 = 0$$

If $x = 10$, $$y = \frac{6}{11} + \frac{7}{11} \times 10 = \frac{76}{11} = 6.91$$

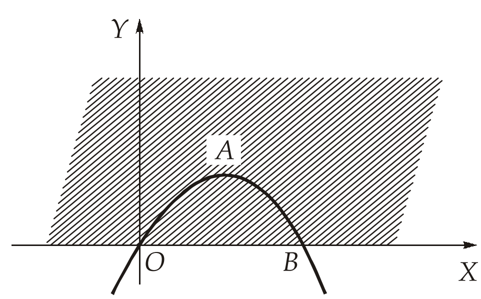
Q.7 (a) Solution:

Case-I: When $m = 0$

In this case, $$y = x - x^2 \quad \text{...(i)}$$

and $$y = 0 \quad \text{...(ii)}$$

are two given curves, $y > 0$ is total region above X-axis. Therefore, area between $y = x - x^2$ and $y = 0$ is area between $y = x - x^2$ and above the X-axis.



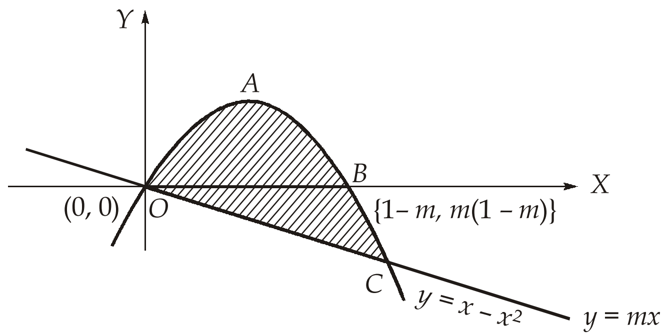
$\therefore$ $$A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right] = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \neq \frac{9}{2}$$

Hence, no solution exists.

Case-II: When $m < 0$

In this case, area between $y = x - x^2$ and $y = mx$ is $OABCO$ and points of intersection are (0, 0) and $\{1 - m, m(1 - m)\}$.

$$\therefore \quad \text{Area of curve } OABCO = \int_0^{1-m} \left[x - x^2 - mx\right] dx$$



$$= \left[(1-m)\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{1-m}$$

$$= \frac{1}{2}(1-m)^3 - \frac{1}{3}(1-m)^3 = \frac{1}{6}(1-m)^3$$

$$\therefore \quad \frac{1}{6}(1-m)^3 = \frac{9}{2}$$

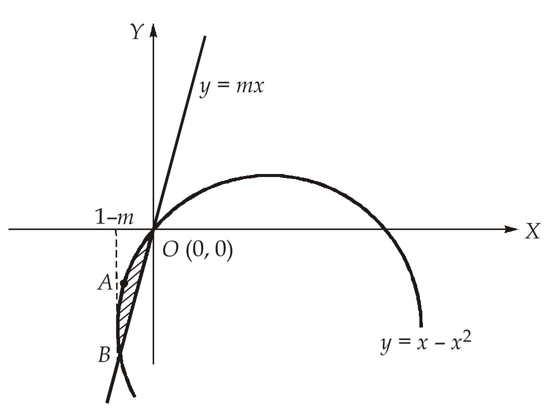
$$\Rightarrow \quad (1 - m)^3 = 27$$

$$\Rightarrow \quad 1 - m = 3$$

$$\Rightarrow \quad m = -2$$

Case III When $m > 0$

In this case, $y = mx$ and $y = x - x^2$ intersect in (0, 0) and $\{(1 - m), m(1 - m)\}$ as shown in figure.



$$\therefore \qquad \text{Area of shaded region} = \int_{1-m}^{0} \left(x - x^2 - mx\right) dx$$

$$= \left[(1-m)\frac{x^2}{2} - \frac{x^3}{3}\right]_{1-m}^{0}$$

$$= -\frac{1}{2}(1-m)(1-m)^2 + \frac{1}{3}(1-m)^3$$

$$= -\frac{1}{6}(1-m)^3$$

$$\Rightarrow \qquad \frac{9}{2} = -\frac{1}{6}(1-m)^3 \qquad \text{[given]}$$

$$\Rightarrow \qquad (1-m)^3 = -27$$

$$\Rightarrow \qquad (1-m) = -3$$

$$\Rightarrow \qquad m = 3 + 1 = 4$$

Q.7 (b) (i) Solution:

$$u = \log(x^3 + y^3 + z^3 - 3xyz) \qquad \text{...(i)}$$

Differentiating (i) partially w.r.t. 'x' we get

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} \qquad \text{...(ii)}$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + 2^3 - 3xyz} \qquad \text{...(iii)}$$

$$\frac{\partial y}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^2 - 3xyz} \qquad \text{...(iv)}$$

On adding (ii), (iii) and (iv), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)}$$

$$= \frac{3}{(x + y + z)}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\frac{3}{x+y+z}$$

$$= -3(x+y+z)^{-2} - 3(x+y+z)^{-2} - 3(x+y+z)^{-2}$$

$$= \frac{-9}{(x+y+z)^2}$$

Q.7 (b) (ii) Solution:

$$x^3 - 2x - 5 = 0$$

Let, $f(x) = x^3 - 2x - 5$ ...(i)

$$f(2) = 8 - 4 - 5 = -1$$

$$f(2.5) = (2.5)^3 - 2 \times 2.5 - 5 = 5.625$$

Since $f(2)$ and $f(2.5)$ are of opposite sign, the root of (1) lies between 2 and 2.5.

$$f'(x) = 3x^2 - 2$$

$$f'(2) = 12 - 2 = 10$$

Let 2 be an approximate root of (1). By Newton-Raphson method

$$a_1 = a - \frac{f(a)}{f'(a)} = 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \frac{-1}{10} = 2.1$$

$$f(2.1) = (2.1)^3 - 2(2.1) - 5 = 9.261 - 4.2 - 5$$

$$= 0.061$$

$$f'(2.1) = 3(2.1)^2 - 2 = 11.23$$

$$a_2 = 2.1 - \frac{f(2.1)}{f'(2.1)} = 2.1 - \frac{0.061}{11.23} = 2.09457$$

$$f(2.09457) = (2.09457)^3 - 2(2.09457) - 5$$

$$= -0.00016$$

$$f'(2.09457) = 3(2.09457)^2 - 2$$

$$= 11.16167$$

$$a_3 = 2.09457 - \frac{f(2.09457)}{f'(2.09457)} = 2.09456$$

Q.7 (c) Solution:

$$F = \frac{\vec{r}}{|\vec{r}|^3} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$$

$$\text{Curl } \vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{(x^2+y^2+z^2)^{3/2}} & \frac{y}{(x^2+y^2+z^2)^{3/2}} & \frac{z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix}$$

$$= \hat{i}\left[\frac{-3}{2}\frac{2yz}{(x^2+y^2+z^2)^{5/2}} + \frac{3}{2}\frac{2yz}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$-\hat{j}\left[\frac{-3}{2}\frac{2xz}{(x^2+y^2+z^2)^{5/2}} - \left(\frac{-3}{2}\right)\frac{2xz}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$-\hat{k}\left[\frac{-3}{2}\frac{2xy}{(x^2+y^2+z^2)^{5/2}} - \left(\frac{-3}{2}\right)\frac{2xy}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$= 0$$

Hence, $\vec{F}$ is irrotational.

$$\vec{F} = \vec{\nabla}\phi, \qquad \text{where } \phi \text{ is called scalar potential}$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{dz}dz$$

$$= \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}}$$

$$\phi = \frac{1}{2}\int \frac{2x\,dx + 2y\,dy + 2z\,dz}{(x^2+y^2+z^2)^{3/2}} = \frac{1}{2}\left(\frac{-2}{1}\right)(x^2+y^2+z^2)^{-1/2}$$

$$\text{scalar potential } (\phi) = \frac{-1}{(x^2+y^2+z^2)^{1/2}} = -\frac{1}{|\vec{r}|}$$

$$\text{Now, Div } \vec{F} = \vec{\nabla}\cdot\vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{(x^2+y^2+z^2)^{3/2} - x\left(\frac{3}{2}\right)(x^2+y^2+z^2)^{1/2}(2x)}{(x^2+y^2+z^2)^3}$$

$$+ \frac{(x^2+y^2+z^2)^{3/2}(1) - y\left(\frac{3}{2}\right)(x^2+y^2+z^2)\frac{1}{2}(2y)}{(x^2+y^2+z^2)^3}$$

$$+ \frac{(x^2+y^2+z^2)^{3/2}(1) - z\left(\frac{3}{2}\right)(x^2+y^2+z^2)\frac{1}{2}(2z)}{(x^2+y^2+z^2)^3}$$

$$= \frac{(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}[x^2+y^2+z^2-3x^2+x^2+y^2+z^2-3y^2+x^2+y^2+z^2-3z^2]$$

$$= 0$$

Hence, $\vec{F}$ is solenoidal.

Q.8 (a) Solution:

(i) Probability of S_1 to be among the eight winners

= (Probability of S_1 being a pair) × (Probability of S_1 winning in the group)

$= 1\times\frac{1}{2} = \frac{1}{2}$ (Since, S_1 is definitely in a group)

(ii) If S_1 and S_2 are in the same pair, then exactly one wins,

If S_1 and S_2 are in two pairs separately, then exact one of S_1 and S_2 will be among the eight winner.

If S_1 wins and S_2 losses or S_1 loses and S_2 wins

Now, the probability of S_1 , S_2 being in the same pair and one wins

= (Probability of S_1 , S_2 being the same pair) × (Probability of anyone winning in the pair)

and the probability of S_1 , S_2 being the same pair

$$= \frac{n(E)}{n(S)}$$

where,

$n(E)$ = The number of ways in which 16 persons can be divided in 8 pairs

$$\therefore \quad n(E) = \frac{(14)!}{(2!)^7 \times 7!} \text{ and } n(S) = \frac{(16)!}{(2!)^8 \cdot 8!}$$

∴ Probability of S_1 and S_2 being in the same pair

$$= \frac{(14)! \times (2!)^8 \times 8!}{(2!)^7 \times 7! \times 16!} = \frac{1}{15}$$

The probability of any one wining in the pairs of S_1 , S_2 = P(certain event) = 1

∴ The pairs of S_1 , S_2 being in two pairs separately and S_1 wins, S_2 loses + The probability of S_1 , S_2 being in two pairs separately and S_1 loses, S_2 wins

$$= \left[1 - \frac{\frac{(14)!}{(2!)^7 \times 7!}}{\frac{(16)!}{(2!)^8 \times 8!}}\right] \times \frac{1}{2} \times \frac{1}{2} + \left[1 - \frac{\frac{(14)!}{(2!)^7 \times 7!}}{\frac{(16)!}{(2!)^8 \times 8!}}\right] \times \frac{1}{2} \times \frac{1}{2}$$

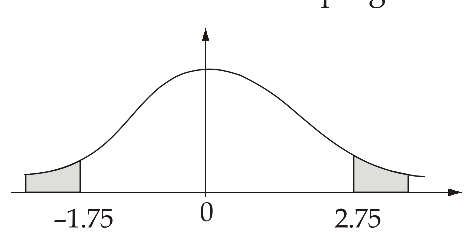
$$= \frac{1}{2} \times \frac{14}{15} \times \frac{(14)!}{14!} = \frac{7}{15}$$

∴ required probability

$$= \frac{1}{15} + \frac{7}{15} = \frac{8}{15}$$

Q.8 (b) (i) Solution:

Tolerance limits of the diameter of non-defective plugs are



0.752 – 0.004 = 0.748 cm

$$0.752 + 0.004 = 0.756 \text{ cm}$$

$$Z = \frac{x-\mu}{\sigma}$$

If $x_1 = 0.748$,

$$Z_1 = \frac{0.748 - 0.7515}{0.002} = -1.75$$

If $x_2 = 0.756$

$$Z_2 = \frac{0.756 - 0.7515}{0.002} = 2.25$$

Area under, $Z_1 = -1.75$ to $Z_2 = 2.25$

$= (\text{Area from } Z = 0 \text{ to } Z_1 = -1.75)$

$+ (\text{Area from } Z = 0 \text{ to } Z_2 = 2.25)$

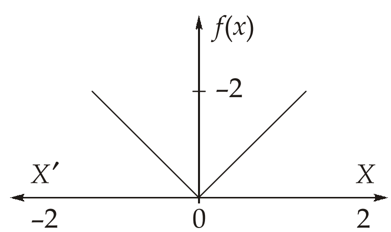
$= 0.4599 + 0.4878 = 0.9477$

Number of plugs likely to be rejected

$= 1000(1 - 0.9477) = 52.3$

Approximately 52 plugs are likely to be rejected.

Q.8 (b) (ii) Solution:



$$f(x) = |x|, \qquad -2 < x < 2$$

$$f(x) = 0, \qquad 0 < x < 2$$

$$= -x, \qquad -2 < x < 0$$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = \frac{1}{2}\int_{0}^{2} x\,dx + \frac{1}{2}\int_{-2}^{0}(-x)dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 + \frac{1}{2}\left[\frac{-x^2}{2}\right]_{-2}^0 = \frac{1}{4}(4-0) + \frac{1}{4}(0+4) = 2$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx = \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx + \frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}dx$$

$$= \frac{1}{2}\left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$$

$$+\frac{1}{2}\left[(-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (-1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_{-2}^0$$

$$= \frac{1}{2}\left[0 + \frac{4}{n^2\pi^2}(-1)^n - \frac{4}{n^2\pi^2}\right] + \frac{1}{2}\left[0 - \frac{4}{n^2\pi^2} + \frac{4}{n^2\pi^2}(-1)^n\right]$$

$$= \frac{1}{2}\frac{4}{n^2\pi^2}\left[(-1)^n - 1 - 1 + (-1)^n\right] = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} \dfrac{8}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$b_n = 0$ as $f(x)$ is even function

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + c_2\cos\frac{2\pi x}{c} + \ldots + b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + \ldots$$

$$= 1 - \frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \frac{\cos\frac{5\pi x}{2}}{5^2} + \ldots\right]$$

Q.8 (c) Solution:

$h = 0.1$

Here, $x_0 = 0, \quad y_0 = 1$

$f(x, y) = x + y$

Now, $K_1 = h_f(x_0, y_0) = 0.1(0 + 1) = 0.1$

$$K_2 = hf\left(x_0 + \frac{h}{2} + y_0 + \frac{K_1}{2}\right) = 0.1f(0 + 0.05, 1 + 0.05)$$

$= 0.1(0.05 + 1.05)$

$= 0.11$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$
$$= 0.1f(0 + 0.05, 1 + 0.055)$$
$$= 0.1105$$
$$K_4 = h_f(x_0 + h, y_0 + K_3)$$
$$= 0.1\, f(0 + 0.1, 1 + 0.1105)$$
$$= 0.1(0.1 + 1.1105)$$
$$= 0.12105$$

According to Runge-Kutta (Fourth order) Formula,

$$y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$
$$y_1 = 1 + \frac{1}{6}(0.1 + 0.22 + 0.221 + 0.12105)$$
$$= 1 + \frac{1}{6}(0.66205) = 1.11034$$

For the second step,

$$x_0 = 0.1, \qquad y_0 = 1.11034,$$
$$y = 0.1$$
$$K_1 = h_f(x_0, y_0)$$
$$= 0.1(0.1 + 1.11034)$$
$$= 0.121034$$
$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$
$$= 0.1f(0.1 + 0.05, 1.11034 + 0.060517)$$
$$= 0.1(0.15 + 1.170857)$$
$$= 0.1320857$$
$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$
$$= 0.1f(0.1 + 0.05, 1.11034 + 0.0660428)$$
$$= 0.1(0.15 + 1.1763828)$$
$$= 0.13263828$$
$$K_4 = hf(x_0 + h, y_0 + K_3)$$

$$= 0.1(0.1 + 0.1, 1.11034 + 0.13263828)$$

$$= 0.1(0.2 + 1.24297828)$$

$$= 0.144297828$$

$$y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$= 1.11034 + \frac{1}{6}[0.121034 + 2 \times 0.1320857 + 2 \times 0.13263828 + 0.144297828]$$

$$= 1.11034 + \frac{1}{6} \times 0.794779788$$

$$= 1.11034 + 0.132463298$$

$$= 1.242803298$$

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