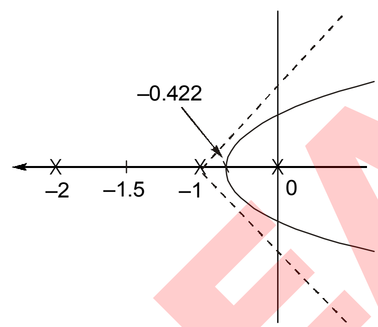
Centroid: It is the interpection point of the asymptotes on the real axis. It may or may not be a part of root locus.

$$\text{Centroid} = \frac{\Sigma \text{ Real part of open loop poles } - \Sigma \text{ Real part of open loop zeros}}{P - Z}$$

$$= \frac{0-1-2}{3} = -1$$

Centroid $\rightarrow (-1, 0)$

Break-away or break-in points: These are those points on whose multiple roots of the characteristic equation occur.



$$s(s^2 + 3s + 2) + K = 0$$

$$K = -(s^3 + 3s^2 + 2s)$$

$$\frac{dK}{ds} = -(3s^2 + 6s + 2) = 0$$

$$s = -0.422, -1.577$$

Now verify the valid break-away point

$$K = 0.234 \text{ (valid) at } s = -0.422$$

$$K = \text{negative (not valid) at } s = -1.577$$

T2. Sol.

$$OLTF = \frac{K}{s(s^2+4s+8)}$$

Poles are at $s_1 = 0$

and $s_{2,\,3} = \dfrac{-4 \pm \sqrt{16-32}}{2} = -2 \pm j2$

There are 3 poles and no zero with root loci, all terminating at infinity.

$$\phi = \frac{(2q+1)180°}{P-Z} = 60°,\ 180°,\ 300° \quad \text{for } q = 0, 1, 2 \text{ (angle of}$$

asymptotes)

$$\text{Centroid} = \frac{0-2+j2-2-j2}{3} = \frac{-4}{3} = -1.33$$

$$1+\frac{K}{s(s^2+4s+8)} = 0$$

$$\Rightarrow \quad K = -s(s^2+4s+8) = -(s^3+4s^2+8s)$$

for break away points, $$\frac{dK}{ds} = 0$$

$$\Rightarrow \quad \frac{dK}{ds} = -(3s^2+8s+8)$$

$$s_{1,2} = -\frac{8\pm\sqrt{64-4\times8\times3}}{2\times3}$$

$$= -\frac{8\pm\sqrt{64-96}}{6} = -1.33\pm j0.943$$

As $\frac{dK}{ds}$ is imaginary, there is no breakaway point from the real axis.

Imaginary axis crossing:

$$\text{Characteristic equation} = s(s^2+4s+8)+K$$
$$= s^3+4s^2+8s+K = 0$$

s^3	1	8
s^2	4	K
s^1	$\frac{32-K}{4}$	
s^0	K	

From Routh-Hurwitz Criteria:

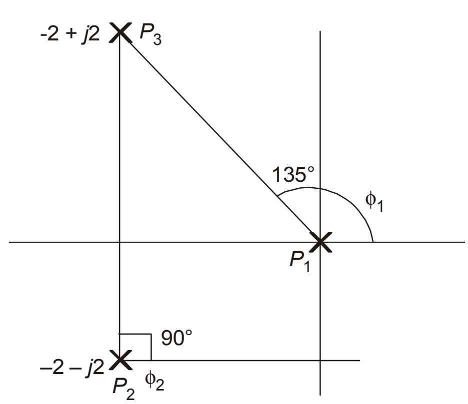
For $K = 32$, the system is marginally stable and beyond $K = 32$ the system becomes unstable.

Hence, $$4s^2 + K = 4s^2 + 32$$

$$s = j2\sqrt{2} = j\omega$$

$$\omega = 2\sqrt{2} = 2.83$$

The root locus cuts the imaginary axis at $\pm j2.83$.



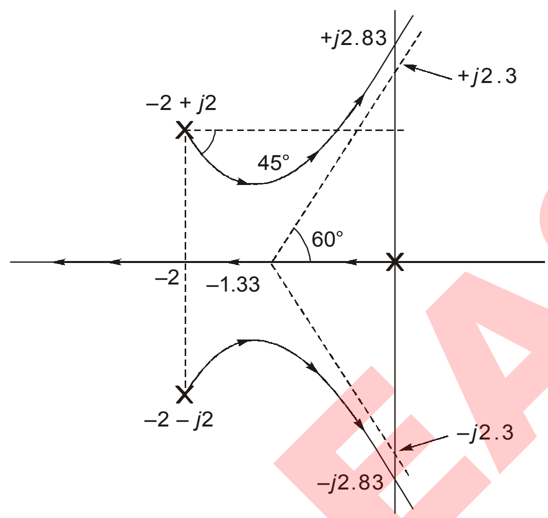
$$\phi_1 = 180° - \tan^{-1}\left(\frac{2}{2}\right) = 135°$$

$$\phi_2 = 90°$$
$$\Sigma\phi_p = \phi_1 + \phi_2 = 135° + 90° = 225°$$
$$\phi = \Sigma\phi_2 - \Sigma\phi_p = 0 - 225° = -225°$$

Angle of departure,

$$\phi_D = 180 + \phi = 180 - 225° = -45°$$

Root locus of the given system:



T3. Sol.

$$G(s)\,H(s) = \frac{K}{s(s+1)(s+4)}$$

Step-1 Number of open loop poles ;

$$P = 3$$

Number of open loop zeros ; $Z = 0$

Number of branches terminating at infinity

$$= P - Z = 3$$

Step-2 Angle of asymptotes

$$\theta = \frac{(2q+1)180°}{P-Z} \quad \text{where } q = 0, 1, 2$$

$$\theta_1 = \frac{180°}{3} = 60°$$

$$\theta_2 = \frac{3 \times 180°}{3} = 180°$$

$$\theta_3 = \frac{5 \times 180°}{3} = 300°$$

Step-3 Centroid $= \dfrac{\Sigma \text{ real part of open loop poles} - \Sigma \text{ real part of open loop zeros}}{P - Z}$

$$= \frac{(-1-4)-(0)}{3-0} = -\frac{5}{3}$$

Step-4 Break away point

$$K + s(s^2 + 5s + 4) = 0$$
$$K = -s^3 - 5s^2 - 4s$$
$$\frac{dK}{ds} = -3s^2 - 10s - 4 = 0$$
$$3s^2 + 10s + 4 = 0$$
$$\Rightarrow \quad s_1, s_2 = -0.4648, -2.8685$$

Valid break-away point will be –0.4648

(i) Routh array table

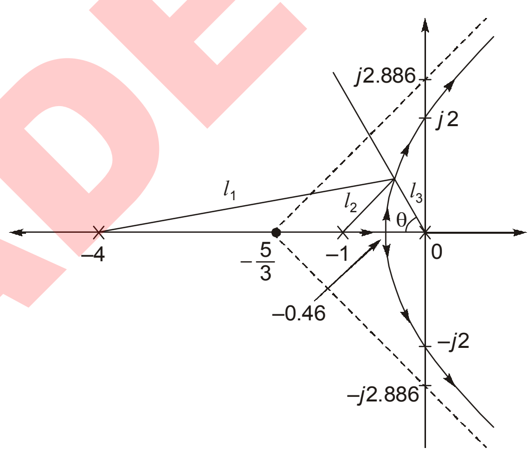
$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 5 & K \\ s^1 & \frac{20-K}{5} & 0 \\ s^0 & 1 & \end{array}$$

For system to be stable

$$20 - K > 0 \quad \Rightarrow \quad K < 20$$

For system to be marginally stable.

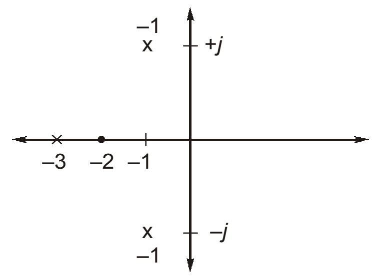
$$K = 20$$
$$A(s) = 5s^2 + 20 = 0$$
$$\Rightarrow \quad s = \pm 2j$$



(ii) To find K,

$$\theta = \cos^{-1}\xi = \cos^{-1}(0.34) = 70.123°$$
$$K = l_1\, l_2\, l_3$$
$$\text{Gain margin } (GM) = \frac{\text{K(Marginal stability)}}{\text{K(desired)}}$$

T4. Sol.



$$\phi_d = 180 - (\phi_p - \phi_z)$$

$$= 180° - \left(180 + \tan^{-1}\left(\frac{1}{2}\right) + 90° - 225°\right)$$

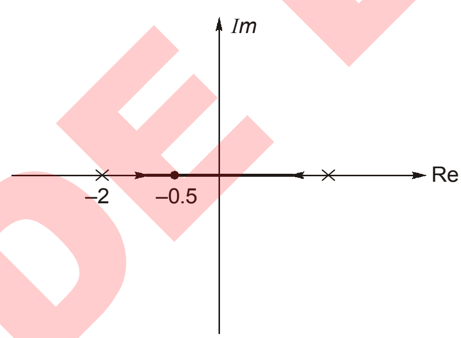
$$= 108.4°$$

T5. Sol.

Given that

$$G(s) = \frac{K}{(s+2)(s-1)}$$

Using root locus method, the break point can be



obtain as

$\Rightarrow$ $$1 + G(s) = 0$$

$$1+\frac{K}{(s+2)(s-1)} = 0$$

or $$K = -(s + 2)(s - 1)$$

$$\frac{dK}{ds} = -2s - 1 = 0$$

or $$s = -0.5$$

To have, both the poles at the same directions

$$|G(s)|_{s = 0.5} = 1$$

$$K = 2.25$$

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5 Frequency Response Analysis

Detailed Explanation of Try Yourself Questions

T1. Sol.

Given, $$G(s) = \frac{K(1+0.5s)(1+as)}{s\left(1+\frac{s}{8}\right)(1+bs)\left(1+\frac{s}{36}\right)}$$

$(1 + as)$ is addition of zero to the transfer function whose contribution in slope = +20 dB/decade or –6 dB/octave.

$(1 + bs)$ is addition of pole to the transfer function whose contribution in slope = –20 dB/decade or –6 dB/octave

Observing the change in the slope at different corner frequencies, we conclude that

$$a = \frac{1}{4} \text{ rad/s and } b = \frac{1}{24} \text{ rad/s}$$

From $\omega = 0.01$ rad/s to $\omega = 8$ rad/s,

slope = –20 dB/decade

Let the vertical length in dB be y

$$\therefore \quad -20 = \left(\frac{0-y}{\log 8 - \log 0.01}\right)$$

or, $$-20 = \frac{y}{\log 8 + 2}$$

or, $y = 58$ dB

Applying $y = mx + C$ at $\omega = 0.01$ rad/s,

we have: $58 = -20 \log 0.01 + C$

or, $C = 58 - 40 = 18$

Now, $C = 20 \log K$

or, $$\log K = \frac{18}{20} = 0.9$$

$$\therefore \qquad K = \log^{-1}(0.9) = (10)^{0.9} = 7.94$$

$$\therefore \qquad \frac{a}{bK} = \frac{\left(\frac{1}{4}\right)}{\left(\frac{1}{24}\right)\times 7.94}$$

$$= \frac{24}{4\times 7.94} = 0.755$$

$$\therefore \qquad \frac{a}{bK} = 0.755$$

T2. Sol.

$$\text{OLTF} = G(s) = \frac{1}{(s+2)^2}$$

For unity feedback system,

$$H(s) = 1$$

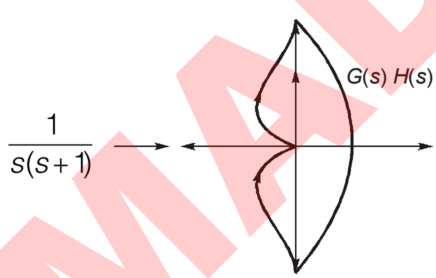
$$\therefore \qquad \text{CLTF} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{1}{(s+2)^2}}{1+\frac{1}{(s+2)^2}}$$

$$= \frac{1}{s^2+4s+5}$$

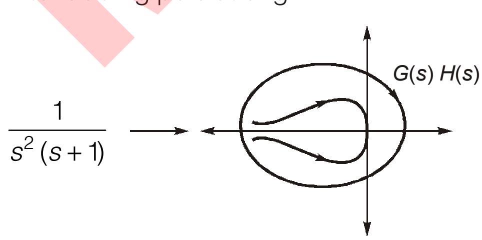
$\therefore$ Close loop poles will be the roots of $s^2 + 4s + 5 = 0$

i.e. $\qquad s = -2 + j$ and $-2 - j$

T3. (b)



After adding pole at origin



So, nyquist plot of a system will rotate by 90° in clockwise direction.

T4. Sol.

For gain margin we have to find

$$G(s)H(s) = \frac{0.75}{s(1+s)(1+0.5s)}$$

Phase over freqeuncy

$$-180° = -90° - \tan^{-1}(\omega) - \tan^{-1}(0.5 - \omega)$$
$$-90° = \tan^{-1}(\omega) + \tan^{-1}(0.5\,\omega)$$
$$\frac{1.5\omega}{1-0.5\omega^2} = \tan(90°)$$
$$0.5\,\omega^2 = 1$$
$$\omega = \sqrt{2}$$

$\therefore$ $$|G(j\omega)H(j\omega)| = \frac{0.75}{\omega\sqrt{1+\omega^2}\sqrt{1+0.25\omega^2}} = \frac{0.75}{\sqrt{2}\sqrt{1+2}\sqrt{1+0.5}} = \frac{1}{4}$$

$\therefore$ $$\text{Gain margin} = 20\log\frac{1}{|G(j\omega)H(j\omega)|}$$
$$= 20\log 4 = 12\text{ dB}$$

T5. Sol.

$$-90° - \tan^{-1}(2\omega) - \tan^{-1}(3\omega) = -180°$$
$$\tan^{-1}(2\omega) + \tan^{-1}(3\omega) = 90°$$
$$\frac{5\omega}{1-6\omega^2} = \tan(90°)$$

$\therefore$ $$1 - 6\,\omega^2 = 0$$
$$\omega = \frac{1}{\sqrt{6}} = 0.41$$

T6. Sol.

The Bode plot is of type zero system
thus steady state error

$$e_{ss} = \frac{1}{1+K_p}$$

Where K_p = propational error constant

K_p = 40 db

or K_p = 100

$\therefore$ $$e_{ss} = \frac{1}{1+100} = \frac{1}{101} = 0.009$$

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6 Controllers and Compensators

Detailed Explanation of Try Yourself Questions

T1. Sol.

The given compensator represents phase lead compensator having maximum phase

$$\phi = \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right)$$

Here,
$$\alpha = \frac{R_2}{R_1+R_2} = \frac{(1/2)\ \Omega}{1+(1/2)\ \Omega} = 0.333$$

$\therefore$
$$\phi = \sin^{-1}\left(\frac{1-0.333}{1+0.333}\right) = \sin^{-1}\left(\frac{0.667}{1.33}\right) = \sin^{-1}(0.5) = 30°$$

T2. (a)

The effect of addition of a zero to a transfer function is providing a phase lead.

T3. Sol.

$$G(s) = \frac{\left(1+\frac{s}{4}\right)}{\left(1+\frac{s}{25}\right)}$$

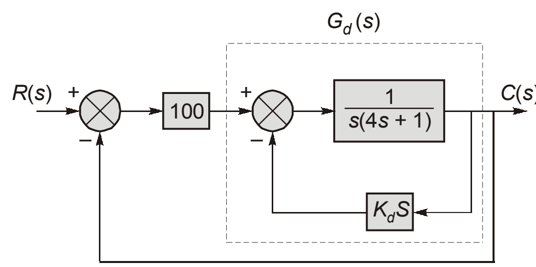
Comparing it with the standard transfer function of phase lead compensator

$$G(s) = \frac{\alpha(1+Ts)}{(1+\alpha Ts)}$$

$$T = \frac{1}{4}, \quad \alpha T = \frac{1}{25}$$

Now, frequency ω_m occurs at
$$= \sqrt{\frac{1}{\alpha T}\cdot\frac{1}{T}} = \sqrt{25\times 4} = 10 \text{ rad/sec.}$$

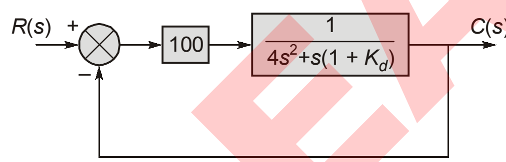
T4. Sol.



Converting the blocked portion into simplified form as

$$G_d(s) = \frac{\frac{1}{s(4s+1)}}{1+\frac{sK_d}{s(4s+1)}} = \frac{1}{4s^2+s(1+K_d)}$$

Now,



Now, simplifying the above block diagram as

$$G(s) = \frac{\frac{100}{4s^2+s(1+K_d)}}{1+\frac{100}{4s^2+s(1+K_d)}}$$

$$= \frac{100}{4s^2+s(1+K_d)+100}$$

$$= \frac{25}{s^2+\frac{s(1+K_d)}{4}+25}$$

Comparing it with standard equation as

$$\omega_n = 5 \text{ rad/sec.}$$

$$2\xi\omega_n = \left(\frac{1+K_d}{4}\right) \qquad \text{Given } \xi = 0.5$$

$$5 = \frac{1+K_d}{4}$$

$\Rightarrow$ $$K_d = 19$$

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7 State Space Analysis

Detailed Explanation of Try Yourself Questions

T1. (b)

T2. Sol.

Given
$$\dot{x}(t) = Ax(t), \quad x(0) = x_0$$

Taking the Laplace transform
$$sX(s) - x(0) = AX(s)$$
$$[sI - A]\,X(s) = x(0)$$
$$X(s) = [sI - A]^{-1}\,x(0)$$
$$x(t) = \mathcal{L}^{-1}[sI - A]^{-1}\,x(0) \qquad \ldots(i)$$

Conditions given are

For
$$x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \; x(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$$

For
$$x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; x(t) = \begin{bmatrix} e^{-t} - e^{-2t} \\ -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Using the linearity property in equation (i)
$$K_1\,x_1(t) = \mathcal{L}^{-1}[sI - A]^{-1}\,x_1(0)\,K_1$$
$$K_2\,x_2(t) = \mathcal{L}^{-1}[sI - A]^{-1}\,x_2(0)\,K_2$$

Using the linearity property as
$$K_1\,x_1(t) + K_2\,x_2(t) = \mathcal{L}^{-1}[sI - A]^{-1}\,[K_1x_1(0) + K_2x_2(0)] \qquad \ldots(ii)$$

Also
$$X_3(s) = [sI - A]^{-1}\,x_3(0)$$

So,
$$K_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + K_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} K_1 + 0K_2 \\ -K_1 + K_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad K_1 = 3$$

$$K_2 = 8$$

So, from equation (ii), we get $x(t)$

$$x(t) = K_1 x_1(t) + K_2 x_2(t)$$

$$= 3\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + 8\begin{bmatrix} e^{-t} - e^{-2t} \\ -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 11e^{-t} - 8e^{-2t} \\ -11e^{-t} + 16e^{-2t} \end{bmatrix}$$

T3. Sol.

Given

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}$$

$$[sI - A] = s^2$$

$$\phi(t) = \mathcal{L}^{-1}[sI - A]^{-1}$$

$$= \frac{1}{s^2}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

T4. (b)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}, \; C = [1 \;\; 1 \;\; 1]$$

Check for controllability:

$$Q_c = [B \; : \; AB \; : \; A^2B]$$

$$= \begin{bmatrix} 0 & 4 & -8 \\ 4 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

For controllable

$$|Q_c| \neq 0$$

Here, $|Q_c| = 4(0) = 0 \;\; \therefore$ Uncontrollable.

Check for observability:

$$Q_o = [C^T \; : \; A^T C^T \; : \; A^{2T} C^T]$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 4 \end{bmatrix}$$

For observable

$$|Q_o| \neq 0$$

Here $|Q_o| = 1 \;\; \therefore$ Observable.

T5. Sol.

$$\text{Characterstic equation} = \left|(sI - A)^{-1}\right|$$

$$= \begin{bmatrix} 2 & -1 \\ 3 & s+5 \end{bmatrix}^{-1}$$

$$= s(s+5) + 3$$

$$= s^2 + 5s + 3$$

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8 Process Control

Detailed Explanation of Try Yourself Questions

T1. Sol.

The open loop transfer function for the primary loop is given by

$$G(s)_{\text{primary}} = K_2 \frac{1}{(s+1)} \cdot \frac{8}{(s+2)(s+4)}$$

Phase cross-over frequency for primary loop is given by

$$-\tan^{-1}\omega_{pc} - \tan^{-1}\frac{\omega_{pc}}{2} - \tan^{-1}\frac{\omega_{pc}}{4} = -180°$$

$$\tan^{-1}\omega_{pc} + \tan^{-1}\frac{\omega_{pc}}{2} + \tan^{-1}\frac{\omega_{pc}}{4} = 180°$$

$$\tan^{-1}\omega_{pc} + \tan^{-1}\left[\frac{\frac{\omega_{pc}}{2} + \frac{\omega_{pc}}{4}}{1 - \frac{\omega_{pc}^2}{8}}\right] = 180°$$

$$\tan^{-1}\omega_{pc} + \tan^{-1}\left[\frac{6\omega_{pc}}{8 - \omega_{pc}^2}\right] = 180°$$

$$\tan^{-1}\left[\frac{\omega_{pc} + \frac{6\omega_{pc}}{8-\omega_{pc}^2}}{1 - \frac{6\omega_{pc}^2}{8-\omega_{pc}^2}}\right] = 180°$$

$$\omega_{pc} + \frac{6\omega_{pc}}{8 - \omega_{pc}^2} = 0$$

$$1 + \frac{6}{8 - \omega_{pc}^2} = 0$$

$$\omega_{pc} = \sqrt{14}$$

We know that magnitude = 1, at ω_{pc} (By polar plot)

$$M = K_2 \frac{1}{\sqrt{1+\omega^2}} \frac{1}{\sqrt{\omega^2+4}} \frac{8}{\sqrt{\omega^2+16}}$$

$$1 = K_2 \frac{1}{\sqrt{1+\omega_{pc}^2}} \frac{1}{\sqrt{\omega_{pc}^2+4}} \frac{8}{\sqrt{\omega_{pc}^2+16}}$$

$$1 = K_2 \frac{1}{\sqrt{15}} \frac{1}{\sqrt{18}} \frac{8}{\sqrt{30}} = \frac{8K_2}{\sqrt{8100}}$$

$$K_2 = \frac{90}{8} = 11.25$$

The open loop transfer function for the secondary loop is given by

$$G_{secondary} = K_1 \frac{1}{s+1}$$

Cross-over frequency for secondary loop

$$-\tan^{-1}\omega_{pc} = -180°$$

$$\omega_{pc} = 0$$

Since there is no cross-over frequency for the secondary loop, so we can use any value of gain K_1 for secondary loop.

$\therefore$ Upper limit of $K_1 = \infty$

Upper limit of $K_2 = 11.25$

T2. Sol.

$$[D(s)\,G_{ff}(s) - Y(s)]\left[\frac{1}{s(s+1)}\right] + D(s) = Y(s)$$

$$D(s)\,G_{ff}(s) + (s^2+s)\,D(s) = (s^2+s+1)\,Y(s)$$

$$\frac{Y(s)}{D(s)} = H(s) = \frac{G_{FF}(s) + s^2 + s}{s^2+s+1}$$

$\because$ $G_{ff}(s) = 1 + s$ [P-D controller]

$\therefore$ $$H(s) = \frac{(s^2+2s+1)}{(s^2+s+1)}$$

$$|H(j\omega)|_{\omega=2} = \frac{5}{\sqrt{9+4}} = \frac{5}{\sqrt{13}}$$

T3. Sol.

Using the value of K_u and P_u, Ziegler and Nichols recommended the following settings for feedback controllers

	K_C	$\tau_{I(\text{min})}$	$\tau_{D(\text{min})}$
P	$K_u/2$	–	–
P-1	$K_u/2.2$	$P_u/1.2$	–
P-1-D	$K_u/1.7$	$P_u/2$	$P_u/8$

Then, the Ziegeer-Nichols setting for the proportional controller is

$$\frac{K_u}{2} = \frac{10}{2} = 5$$

T4. Sol.

Open loop transfer function is given by

$$G(s) = \left[K_p + \frac{K_I}{s}\right]\frac{1}{(s+2)(s+10)} = \left[\frac{K_p s + K_I}{s}\right]\frac{1}{(s+2)(s+10)}$$

Ch: equation is given by

$$1 + G(s) = 0$$
$$s(s+2)(s+10) + K_p s + K_I = 0$$
$$s^3 + 12s^2 + (20 + K_p)s + K_I = 0$$

By R-H critieria

$$\begin{array}{c|cc} s^3 & 1 & 20 + K_p \\ s^2 & 12 & K_I \\ s^1 & \dfrac{12(20+K_p) - K_I}{12} & 0 \\ s^0 & K_I & \end{array}$$

For stable system, $\boxed{K_I > 0}$

$$12(20 + K_p) \geq 0$$
$$240 + 20K_p - K_I \geq 0$$
$$1240 + 20\,K_p \geq K_I$$
$$K_p \geq \frac{K_I - 240}{20}$$

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