MPSC 2019

Maharashtra Public Service Commission

Assistant Engineer Examination

Civil Engineering

Structural Analysis

Well Illustrated **Theory** *with*

Solved Examples and **Practice Questions**

MADE EASY Publications

Structural Analysis

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CHAPTER 01

Stability and Indeterminacy

1.1 Support System

1.1.1 2-D Supports

(a) Fixed Support

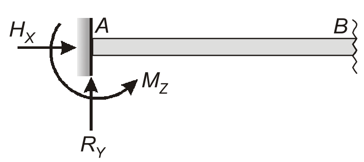


Fig. 1.1 (i) Number of reaction = 3

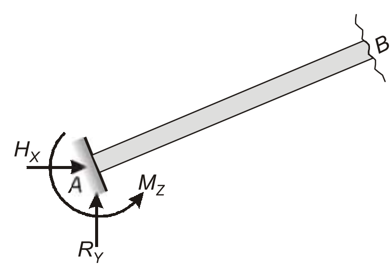


Fig. 1.1 (ii) Number of reactions = 3

At 2-D fixed support, there can be three reactions:

(*i*) one vertical reaction (R_y)
(*ii*) one horizontal reaction (H_x)
(*iii*) one moment reaction (M_z)

(b) Hinge Support

Hinge support is represented by the symbol .

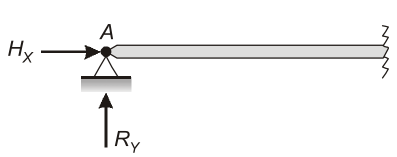


Fig. 1.2 (i) Number of reactions = 2

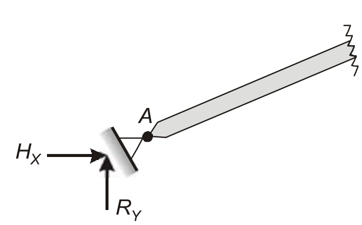


Fig. 1.2 (ii) Number of reactions = 2

At hinged support, there can be two reactions:

(*i*) one horizontal reaction (H_x)
(*ii*) one vertical reaction (R_y)

(c) Roller Support

Roller support is represented by the symbol  or .

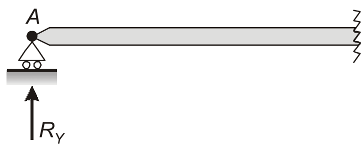


***Fig. 1.3 (i)** Number of reactions = 1*

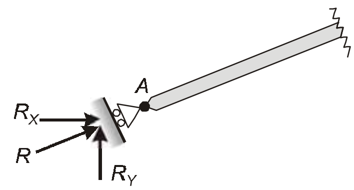


***Fig. 1.3 (ii)** Number of reactions = 1*

At roller support there can be only one externally independent reaction which is normal to the contact surface.

(d) Guided Roller Support

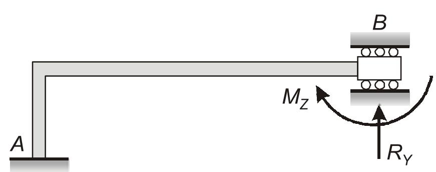


***Fig. 1.4** Number of reactions = 2*

At guided roller supports there can be two reactions:

(*i*) one vertical reaction (R_y)

(*ii*) one moment reaction (M_z)

1.1.2 2-D Internal Joints

(a) Internal Hinge

At internal hinge bending moment will be zero.

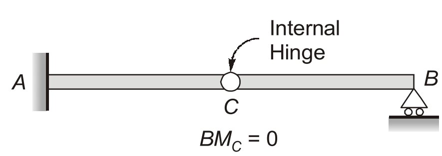


Fig. 1.5

NOTE: An internal hinge provides one additional equilibrium equation for structures.

(b) Internal Roller

At internal roller either axially force or shear force will be zero.

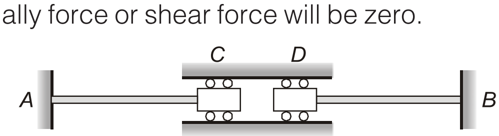


Fig. 1.6

In fig. 1.6, axially force at *C* and *D* is zero.

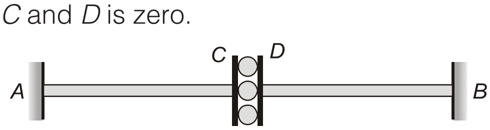


Fig. 1.7

In fig. 1.7, shear force at C and D will be zero i.e., $S_C = S_D = 0$

(c) Internal Link

If any member is connected by hinges at its end and subjected to no external loading in between then it can be termed as internal link and carry axial force only.

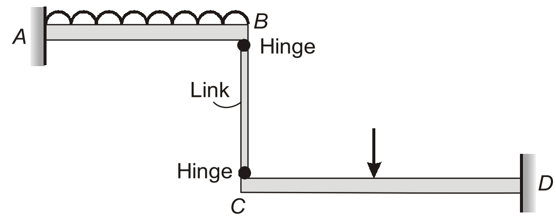


Fig. 1.8

Here BC is a link, link BC carry only axial force

Also $BM_B = 0$ and $BM_C = 0$

NOTE: Internal release also provides additional equation for analysis of structure.

1.1.3 3-D Supports

(a) Fixed Support

At 3-D fixed support there can be six reactions:

(i) three reactions R_x, R_y and R_z

(ii) three moment reactions M_x, M_y and M_z

The fixed support are also called *Built-in support*.

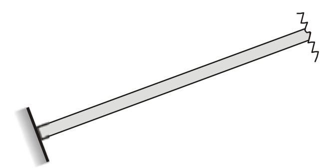


Fig. 1.9: *Number of reactions* = 6

(b) 3-D Hinged Support

At 3-D hinged support there can be three reactions

(i) R_x

(ii) R_y

(ii) R_z

The 3-D hinged support is also called **'ball and socket joint'**.

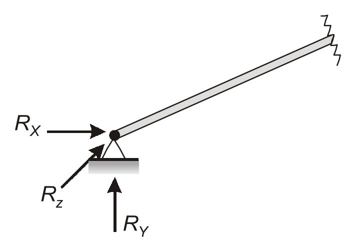


Fig. 1.10 *Number of reactions* = 3

(c) Roller Support

At 3-D roller support there can be only one externally independent reaction which is perpendicular to the contact surface

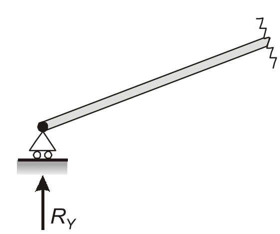


Fig. 1.11 Number of reactions = 1

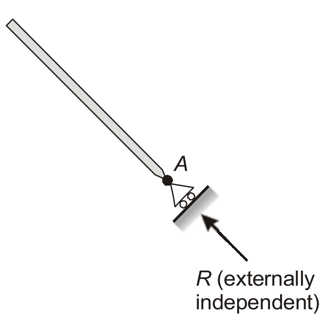


Fig. 1.12 (i)

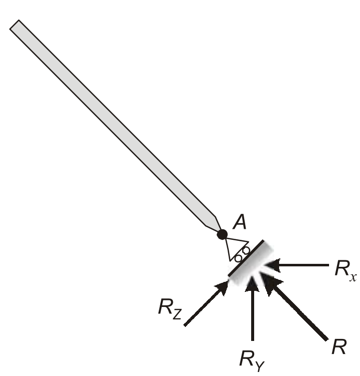


Fig. 1.12 (ii)

in figure 1.12.(ii), reactions at roller support A, R_x, R_y and R_z are externally dependent reactions which depends on reaction R.

1.2 Structure

1.2.1 Elements of Structure

Some of the major elements of structure by which structures are fabricated are as follows:

(a) Beams: Beams are structural members which is predominantly subjected to bending. On the basic of support system beams can be classified as:

(*i*) Simply supported beam

Fig. 1.13

(*ii*) Cantilever beam

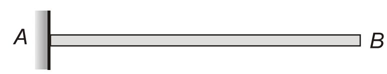


Fig. 1.14

(*iii*) Propped cantilever

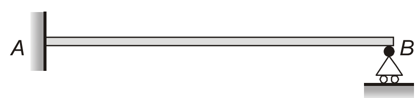


Fig. 1.15

(*iv*) Fixed beam

Fig. 1.16

(*v*) Continuous beam

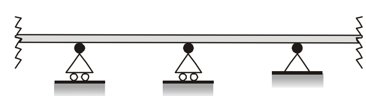


Fig. 1.17

(b) Columns: A column is a vertical compression member which is slender and straight. Generally columns are subjected to axial compression and bending moment as shown in figure.

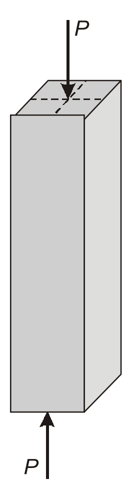


Fig. 1.18 (i)

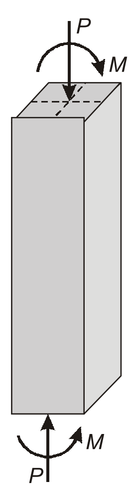


Fig. 1.18 (ii)

(c) Tie Members: Tie members are tension members of trusses and frame, which are subjected to axial tensile force. (Figure : 1.19)

Fig. 1.19 *Tie Rod*

1.2.2 Types of Structures

(a) Trusses: A truss is constructed from pin jointed slender members, usually arranged in triangular manner. In trusses, loads are applied on joints due to which each member of truss subjected to only axial forces i.e., either axial compression or axial tension. Generally trusses are used when span of structure is large.

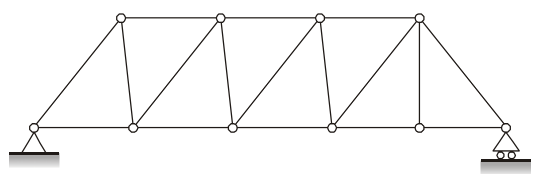


Fig. 1.20 *Truss*

(b) Frames: A frame is constructed from either pin jointed or fixed jointed beam and columns. Generally loads are applied on beams and this loading causes axial force, shear force and bending to the members of frame.

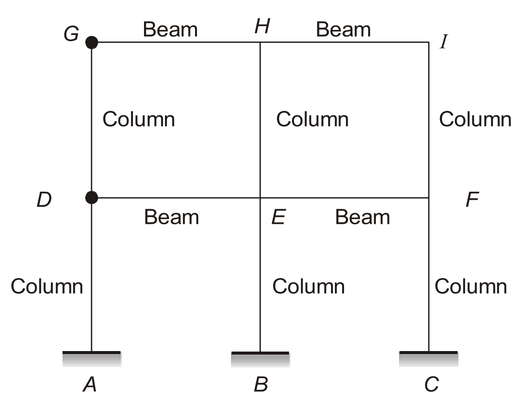


Fig. 1.21 *Frames*

(c) Arches: Arches are used in bridges, dome roof, auditorium, where span of structures are relatively more due to external loading, Arch can be subjected to axial compression, shear force or bending moment.

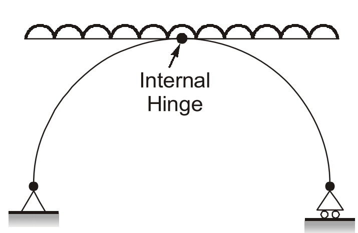


Fig. 1.22 (i) *Three Hinge Arch*

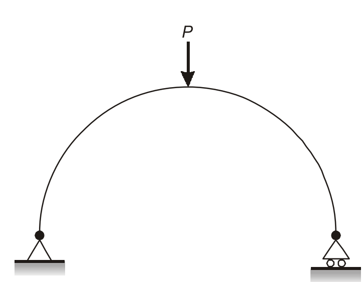


Fig. 1.22 (ii) *Two Hinge Arch*

(d) Cables: Cables are used to support long span bridges. Cables are flexible members and due to external loading it is subjected to axial tension only.

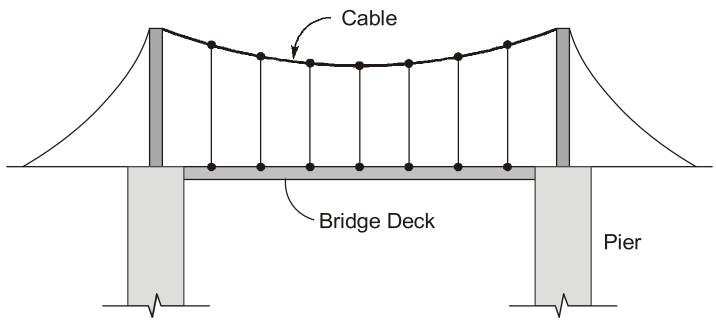


Fig. 1.23 *Cable and Bridge*

1.3 Types of Loading

(a) Point load: A point load is considered to be acting at a point. It is also called concentrated load. In actual practice point loads are distributed load which are distributed over very small area.

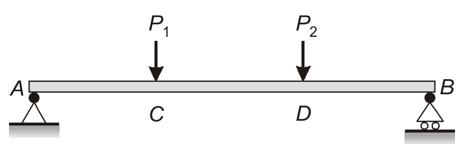


Fig. 1.24 *Point Load*

(b) Distributed loads: Distributed loads are those loads, which acts over some measurable area. Distributed loads are measured by the intensity of loading per unit length along the beam.

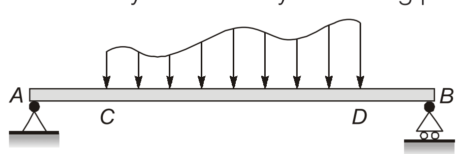


Fig. 1.25 *Distributed Loads*

(c) Uniformly distributed loads: Uniformly distributed loads are those distributed loads which have uniform intensity of loading over the area.

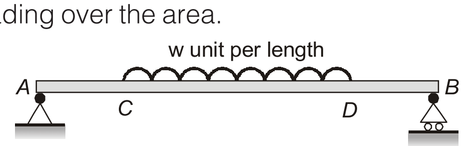


Fig. 1.26 *Uniformly Distributed Loads*

(d) Uniformly varying loads : A uniformly varying load, commonly abbreviated as UVL, is the one in which the intensity of loading varies from one end to other. For example, intensity is zero at one end and *w* at other end.

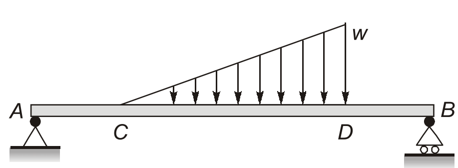


Fig. 1.27 *Uniformly Varying Loads*

(e) Couple : A system of forces with resultant moment, but no resultant force is called couple. It is statically equivalent to force times the offset distance.

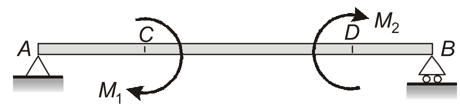


Fig. 1.28 *Couple*

1.4 Stability of Structures

Structural stability is the major concern of the structural designer. To ensure the stability, a structure must have enough support reaction along with proper arrangement of members. The overall stability of structures can be divided into

(*i*) External stability
(*ii*) Internal stability

1.4.1 External Stability

(a) 2-D Structures: For stability of 2-D structures there should be no rigid body movement of structure due to loading so, it should have support in x-direction, y-direction and no rotation in x-y plane. So there should be enough reactions to restrain the rigid body motion.

For stability of 2-D structures, following three conditions of static equilibrium should be satisfied.

(*i*) $\Sigma F_x = 0$ (To prevent Δ_x)
(*ii*) $\Sigma F_y = 0$ (To prevent Δ_y)
(*iii*) $\Sigma M_z = 0$ (To prevent θ_z)

For stability in 2-D structures following conditions also be satisfied:

(*i*) There should be minimum three number of externally independent support reaction.
(*ii*) All reactions should not be parallel, otherwise linearly unstability will set up.

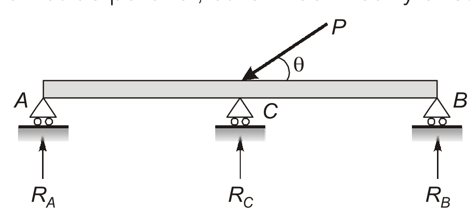


Fig. 1.29 *Unstable*

(*iii*) All reactions should not be linearly concurrent otherwise rotational unstability will setup.

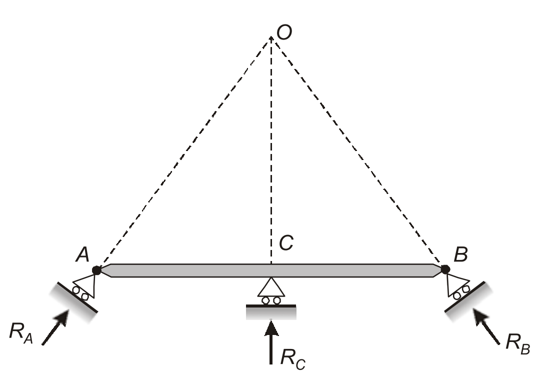


Fig. 1.30 (i) *Unstable*

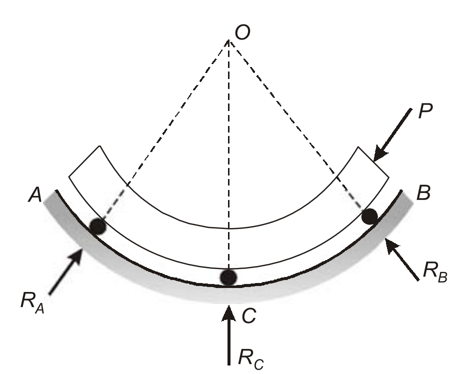


Fig. 1.30 (ii) *Unstable*

(iv) Reactions should be non-trival i.e. there should be enough magnitude and enough difference between them.

(b) 3-D Structures: In case of 3-D structures, there should be a minimum of six independent external reactions to prevent rigid body displacement of structure. The displacement to be prevented are: Δ_x, Δ_y, Δ_z, θ_x, θ_y and θ_z. Therefore, there will be six equation of static equilibrium.

(i) $\Sigma F_x = 0$ (ii) $\Sigma F_y = 0$ (iii) $\Sigma F_z = 0$

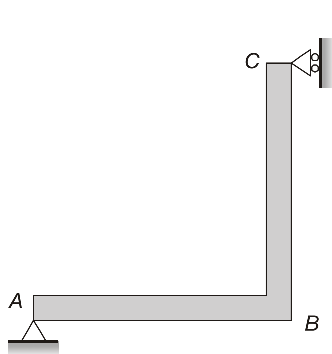
(iv) $\Sigma M_x = 0$ (v) $\Sigma M_y = 0$ (vi) $\Sigma M_y = 0$

For stability in 3-D structures, all the reactions should be non-coplanar, non-concurrent and non-parallel.

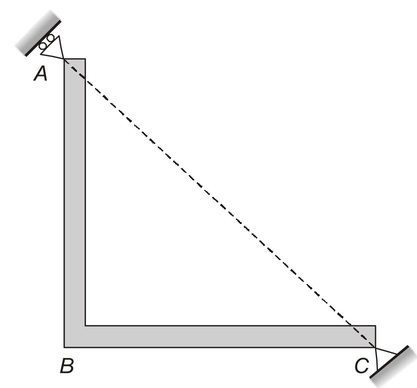
NOTE: If a structure is constructed from elastic members then small elastic displacement may be permitted but small rigid body displacement will not be permitted.

Example-1.1 Which one of the following structures is stable?

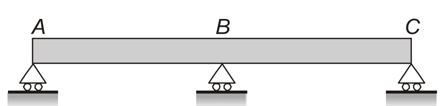
(a)



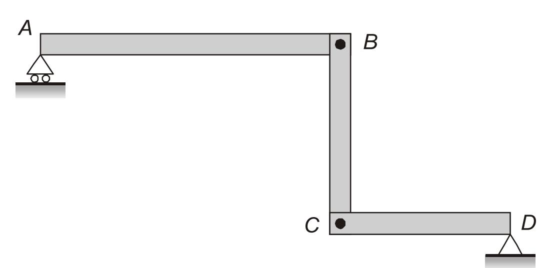
(b)



(c)

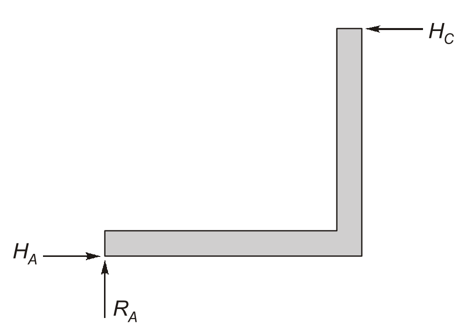


(d)

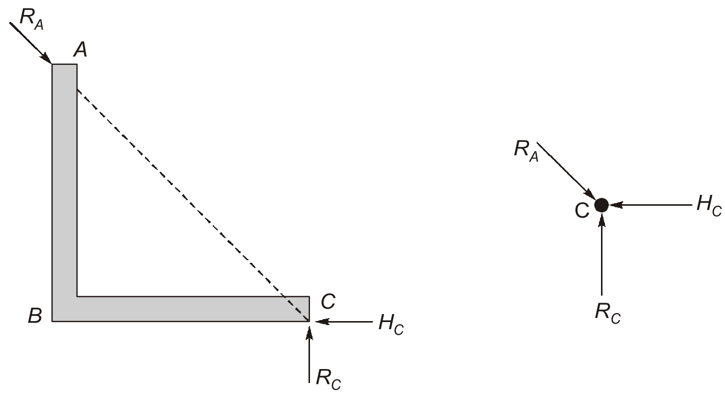


Solution :(a)

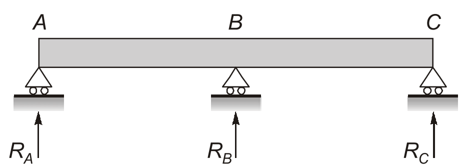
Member (a) is stable, since reactions are non-parallel and non-concurrent.



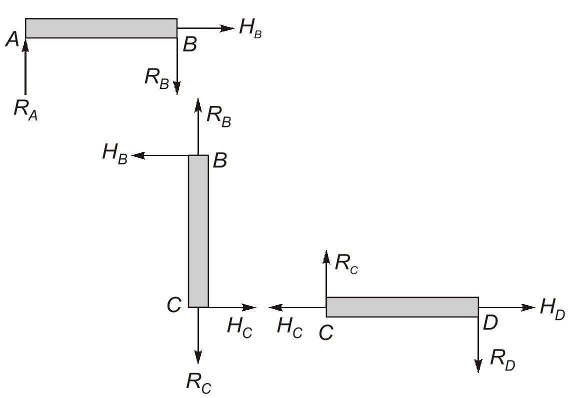
Member (b) is unstable since all the reactions are concurrent at C.



Beam (c) is unstable, since all three reactions are parallel.



Structure (d) is unstable, since the member AB can move horizontally without any restrain. i.e. $\Sigma F_x \neq 0$



1.4.2 Internal Stability

For the internal stability, no part of the structure can move rigidly relative to the other part so that geometry of the structure is preserved, however small elastic deformations are permitted. To preserve geometry, enough number of members and their adequate arrangement is required. For the geometric stability, there should not be any condition of mechanism. Mechanism is formed when there are three collinear hinges, hence to preserve geometric stability there should not be three collinear hinges.

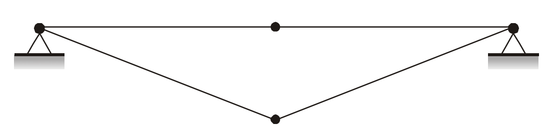


Fig. 1.31

For 2-D truss the minimum number of members needed for geometric stability are:

$$m = 2j - 3$$

and for 3-D truss,

$$m = 3j - 6$$

where, j = Number of joint in truss
m = Member required for geometrical stability.
All the members should be arranged in such a way that truss can be divided into triangular blocks. i.e. no rectangular or polygonal blocks.

Hence, for overall geometrical stability of truss:

(i) Minimum number of member should be present

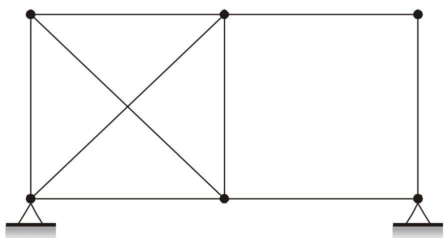
$$m = 2j - 3 \quad \text{(2-D truss)}$$

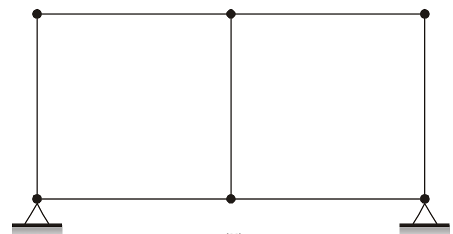
and

$$m = 3j - 6 \quad \text{(3-D truss)}$$

(ii) There should be no condition of mechanism i.e. no three collinear hinges.

Example-1.2 Check geometrical stability for given trusses.


(i)


(ii)

Solution:

(i) In case (i), arrangement of members is not adequate, hence right panel is unstable and left panel is over stiff. For geometric stability, all panels of truss should be stable so given truss is geometrical unstable.

For right panel: $j = 4$
Number of member present, $m = 4$
But minimum number of member needed $= 2j - 3 = 2 \times 4 - 3 = 5$
Hence Right panel is deficient.
For left panel: $j = 4$
Number of member present, $m = 6$
But minimum number of member needed $= 2j - 3 = 2 \times 4 - 3 = 5$
Hence left panel is over stiff.

(ii) $j = 6$
Number of members present, $m = 7$
But minimum number of member needed $= 2j - 3 = 2 \times 6 - 3 = 9$
Hence, above truss is geometrically unstable and it can be called 'deficient structure'.
$\therefore$ Number of deficiency $= 2$

1.5 Statically Determinate and Indeterminate Structures

1.5.1 Statically Determinate Structures

A structure is said to be determinate if conditions of static equilibrium are sufficient to analyse the structure.

- In determinate structures, bending moment and shear force are independent of properties of material and cross-sectional area.
- No stresses are induced due to temperature changes.
- No stresses are induced due to lack of fit and support settlement.

1.5.2 Statically Indeterminate Structures

A structure is said to be statically indeterminate if conditions of static equilibrium are not sufficient to analyse the structure. To analyse these structures, additional compatibility conditions are required.

- In indeterminate structures, bending moment and shear force depends upon the properties of material and cross-sectional area.
- Stresses are induced due to temperature variation.
- Stresses are induced due to lack of fit and support settlement.

1.6 Degree of Indeterminacy

The degree of indeterminacy can be divided into:

1. Static indeterminacy, which can be classified as
   (a) external indeterminacy
   (b) internal indeterminacy
2. Kinematic indeterminacy

1.6.1 Static Indeterminacy

Those structures which can not be analyse using equations of static equilibrium alone are called indeterminate structures or hyper static structures. To analyse these structures extra equation are required which is called compatibility equation.

(a) External Static Indeterminacy (D_{Se}):

It is related to support system of the structure. External static indeterminacy is equal to number of independent external reactions in excess to available equilibrium condition for static equilibrium.

$$D_{Se} = r_e - r$$

where, r_e = Total number of independent support reaction

r = Total number of available equations of static equilibrium

$= 3$ [2-D] ... [2-D]

$= 6$ [3-D] ... [3-D]

Case-1: (2-D beam subjected to general loading)

Here, $r_e = 6$

$r = 3$ ... (2-D)

Therefore, $D_{Se} = r_e - 3$

$D_{Se} = 6 - 3 = 3$

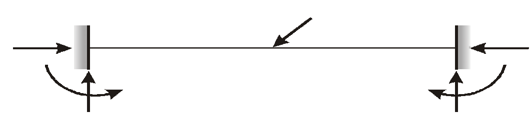


Fig. 1.34

For general loading system, a fixed beam is statically indeterminate to 3rd degree.

However for vertical loading system.

Case-2: (2-D beam vertical loading)

$r_e = 4$

and equations of static equilibrium available,

$r = 2$

therefore, $D_{Se} = r_e - r$

$= 4 - 2 = 2$

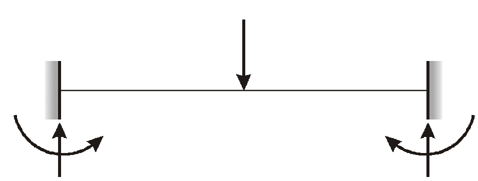


Fig. 1.35

Here beam indeterminate to 2nd degree.

Hence, for general loading, the external indeterminacy is given by

$D_{Se} = r_e - 3$ [For 2-D]

and $D_{Se} = r_e - 6$ [For 3-D]

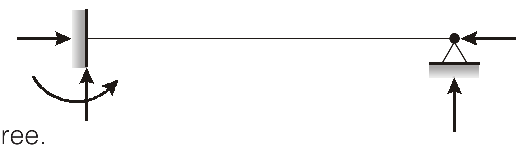
Example-1.3 For the structure shown in figure. Determine degree of external static indeterminacy (D_{Se})

Solution:

$$r_e = 5$$

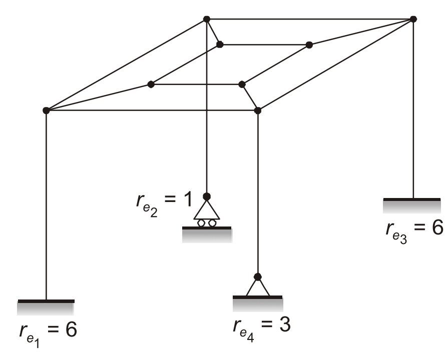
For general loading,

$$D_{Se} = r_e - 3$$
$$= 5 - 3 = 2$$



Hence given beam is externally indeterminate to 2nd degree.

Example-1.4 For the space frame shown in figure determine D_{Se}.



Solution:

Total

$$r_e = r_{e1} + r_{e2} + r_{e3} + r_{e4}$$
$$= 6 + 1 + 3 + 6$$
$$= 16$$

For general loading,

$$D_{Se} = r_e - 6 \text{ ..... (3-D)}$$
$$= 16 - 6 = 10$$

Since all reactions are nonparallel and nonconcurrent, hence given frame is stable and indeterminate to 10th degree.

(b) Internal Static Indeterminacy (D_{si}):

Case-I: Pin jointed plane frame (2-D Truss):

In trusses, all joints are hinged and loading is applied at joint only, the self weight of members are neglected. Hence all member of truss will carry only axial force either tension or compression. If there are m members in the truss, then there will be m internal member force (axial force in each member). At each joint in the truss, there are two equilibrium conditions i.e. $\Sigma F_x = 0$ and $\Sigma F_y = 0$. Let there are j number of joint. Hence total equilibrium conditions available on all joint will be $2j$, out of $2j$ equilibrium conditions, three equations are used to determine external support reactions. Hence net available equations to determine internal reactions will be $2j - 3$.

Therefore, D_{Si} = Total number of internal reactions – Available equation of equilibrium

$$D_{Si} = m - (2j - 3)$$

if $D_{Si} = 0$ — Truss is internally determinate. Such trusses are called perfect trusses

if $D_{Si} > 0$ — It will be internally indeterminate and over stiff

if $D_{Si} < 0$ — Internally deficient and geometrically unstable

Case-II: 3-D Truss (Pin-jointed space frame)

In 3-D truss, each member is having one internal force i.e. axial force but each joint has three condition of equilibrium i.e. $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma F_z = 0$. Therefore, total condition of equilibrium at j number of joint will be $3j$. Out of $3j$ equilibrium conditions six conditions are used to find external support reactions. Hence,

D_{Se} = Total number of internal reactions – Available equation of equilibrium

$$D_{Se} = m - (3j - 6)$$

where,

m = total number of members

j = total number of joints

Case-III: (2-D and 3-D Rigid Frames)

In rigid frames, internal indeterminacy will not exist if it forms an open configuration like a tree. To check internal indeterminacy following thumb rule may be applied.

(*i*) If structure is internally determinate then it is impossible to make a cut anywhere in structure without splitting the structure into two free bodies.

(*ii*) In case of internally determinate structure, it is impossible to return back at same point without retracing the path. It mean internally determinate structures do not have any cyclic loop.

In two dimensional (2-D) rigid frame, each member has three-internal forces i.e. R_x, R_y and M_z and in 3-D rigid frame each member has six internal force i.e. R_x, R_y, R_z, M_x and M_z. It means each closed loop in 2-D has three internal indeterminacy and each closed loop in 3-D has six internal indeterminacy. Hence

For 2-D rigid frames, $D_{Si} = 3C$

For 3-D rigid frames, $D_{Si} = 6C$

where, C = Number of closed loop

In above analysis all the joints are considered rigid. If some of the joints are hybrid (hinged) then some of the internal reactions will released. Hence D_{Si} will reduced.

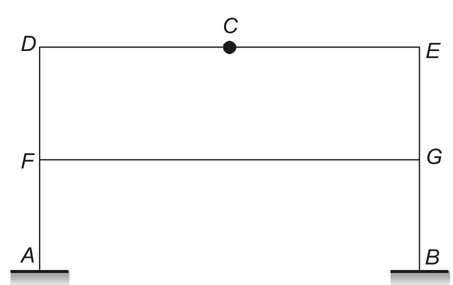
Therefore, $D_{Si} = 3C - r_r$ For 2-D

and $D_{Si} = 6C - r_r$ For 3-D

where, r_r = Total number of internal reaction released.

For example:

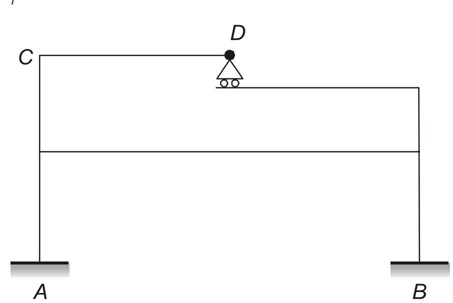
(i)



at C, $M_c = 0$, hence, one internal force (M_c) is released

$\therefore$ $r_r = 1$

(ii)



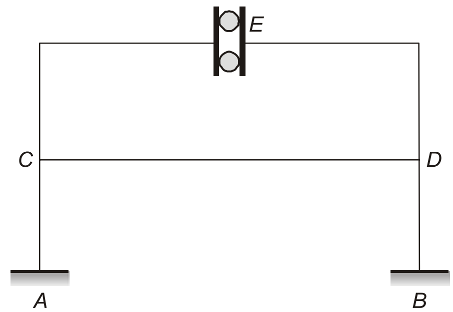
At D, two internal forces are released

(a) Axial Force (AF)

(b) Bending Moment (BM)

$\therefore$ $r_r = 2$

(iii)



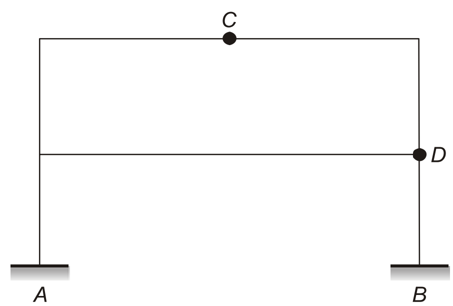
At E, Shear Force (SF) is released

$\therefore \quad r_r = 1$

The number of released forces (r_r) depends upon number of members meeting at hybrid joints.

For example:

(iv)



At C, two members meets. Hence one internal reaction will released.

$\therefore \quad r_r = 1$

At D, three members meets. Hence, two internal reactions will released.

$\therefore \quad r_r = 2$

Hence, total, $\quad r_r = 1 + 2 = 3$

We can generalize internal reaction released as follows:

For plane structures, $\quad r_r = \Sigma(m' - 1)$ (2-D frame)

For space structures, $\quad r_r = \Sigma\, 3(m' - 1)$

where, $\quad m'$ = Number of members meeting at hybrid joint

Hence D_{Si} can be written as

$$D_{Si} = 3C - \Sigma(m' - 1) \quad ..(2\text{-}D)$$

$$= 6C - \Sigma 3(m' - 1) \quad ...(3\text{-}D)$$

where, $\quad C$ = Number of closed loops

m' = Number of members meeting at hybrid joint

Overall Degree of Static Indeterminacy (D_S)

$$D_S = \text{External static indeterminacy} + \text{Internal static indeterminacy}$$

$$D_S = D_{Se} + D_{Si}$$

Alternative Approach to Find D_s

(a) Plane Truss (2-D Truss)

$$D_S = \text{Total unknown forces (External + Internal)} - \text{Total equation of equilibrium available}$$

$$= (m + r_e) - 2j$$

where, $\quad m$ = Number of members (Number of internal reactions)

r_e = Number of external reactions

j = Number of joints

if $D_S = 0$ Truss is statically determinate

$D_S > 0$ Truss is statically indeterminate

$D_S < 0$ Truss is statically unstable

(b) Space Truss (3-D Truss)

$$D_S = (m + r_e) - 3j$$

(c) 2-D Rigid Frames

$D_S = (3m + r_e) - 3j$ (When all joints are rigid)

$D_S = (3m + r_e) - 3j - r_r$ (When some joints are hybrid)

(d) 3-D Rigid Frames

$D_S = (6m + r_e) - 6j$ (When all joints are rigid)

$D_S = (6m + r_e) - 6j - r_r$ (When some joints are hybrid)

Note: Static indeterminacy for frames:

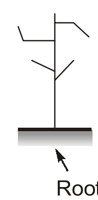
- Frames are rigid joined structure hence all joints are made rigid by providing extra restraint (r_r) the structure is then cut to make open tree like structure.
- Number of trees to be made to be equal to number of support, tree can't have closed loop branch and non of the branches should fall during cutting and also, tree should have only one root.

$D_s = 3C - r_r \rightarrow$ for plane frame

$= 6C - r_r \rightarrow$ for space frame

r_r = Restrain required to make rigid

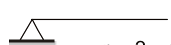
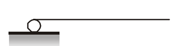
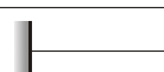
C = Number of cut required to make open tree like structure



- Number of restrained required to make support rigid (r_r)

= [Number of support rk^n had the support being fixed] – [Existing number of support reactions]

Ex:

2D	3D
 $r_r = 3 - 2 = 1$	$r_r = 6 - 3 = 3$
 $r_r = 3 - 1 = 2$	$r_r = 6 - 1 = 5$
 $r_r = 0$	$r_r = 0$

Ex: (1)

Joint A and I have to made rigid before cutting.

Restrain required at joint $A = 3 - 2 = 1$

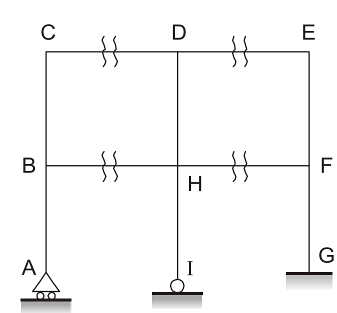
Restrain required at joint $I = 3 - 1 = 2$

$\therefore$ Total restrain required $= 2 + 1 = 3$

$C = 4$

$\therefore$ $D_s = 3C - r_r = 3 \times 4 - 3 = 9$

$D_{se} = 6 - 3 = 3$

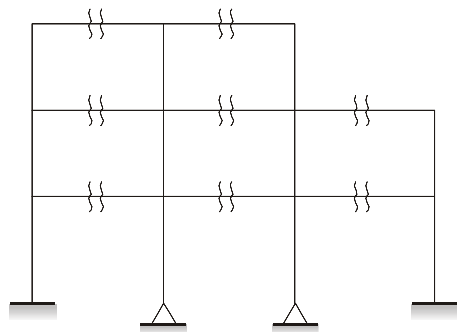


$$D_{si} = D_s - D_{se}$$
$$= 9 - 3 = 6$$

'or' by loop method

$$D_{si} = 3 \times C \text{ (Number of closed loop)}$$
$$D_{si} = 3 \times 2 = 6$$

(2)



$$C = 8;\ r_r = 1 + 1 = 2$$
$$D_{se} = 3C - r_r$$
$$= 3 \times 8 - 2 = 22$$

'or' by loop method

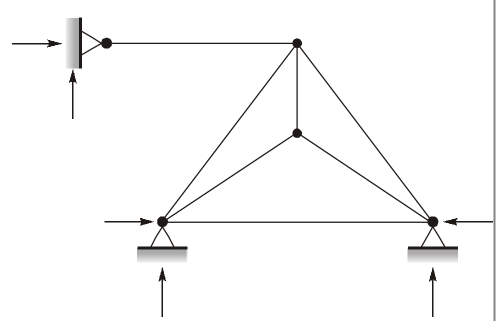
$$D_{si} = 3 \times 5 = 15$$
$$D_{se} = 10 - 3 = 7$$

$\therefore$ $$D_s = D_{si} + D_{se}$$
$$= 15 + 7 = 22$$

Example-1.5 For 2-D truss shown in figure, find degree of static indeterminacy.

Solution:

First approach:



$$D_{Se} = r_e - 3 \text{ (For general loading)}$$
$$= 6 - 3 = 3$$
$$D_{Si} = m - (2j - 3)$$
$$= 7 - (2 \times 5 - 3) = 0$$

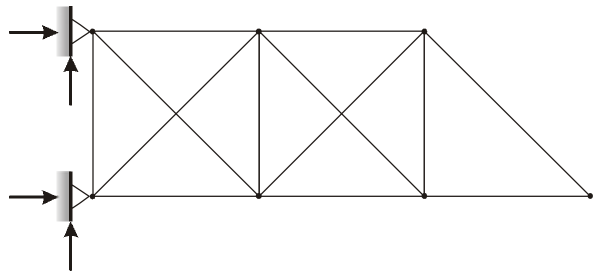
$\therefore$ Degree of static indeterminacy,

$$D_S = D_{Si} + D_{Se}$$
$$D_S = 3 + 0 = 3$$

Second approach:

$$D_S = m + r_e - 2j$$
$$= 7 + 6 - 2 \times 5$$
$$= 13 - 10 = 3$$

Example-1.6 What is the total degree of static indeterminacy (both internal and external) of the cantilever plane truss shown in the figure below?



(a) 2 (b) 3
(c) 4 (d) 5

Solution:(b)

$$m = 13$$
$$j = 7$$
$$r_e = 4$$

First approach:

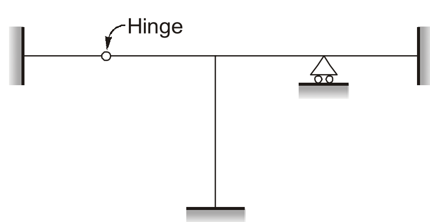
$$D_{Se} = r_e - 3 = 4 - 3 = 1$$
$$D_{Si} = M - (2j - 3)$$
$$= 13 - (2 \times 7 - 3) = 2$$

$\therefore$
$$D_S = D_{Se} + D_{Si}$$
$$D_S = 1 + 2 = 3$$

Second approach:

$$D_S = m + r_e - 2j = 13 + 4 - 2 \times 7 = 3$$

Example-1.7 The degree of static indeterminacy for the rigid frame as shown below is



(a) 6 (b) 4
(c) 8 (d) 10

Solution:(a)

First approach:

$$m = 5$$
$$j = 6$$
$$r_e = 10$$
$$r_r = 1$$
$$D_{Se} = r_e - 3 = 10 - 3 = 7$$
$$D_{Si} = 3C - r_r = 0 - 1 = -1$$
$$D_S = D_{Se} + D_{Si}$$
$$D_S = 7 - 1 = 6$$

Second approach:

$$D_S = 3m + r_e - 3j - r_r$$
$$= 3 \times 5 + 10 - 3 \times 6 - 1 = 6$$