Electronics Engineering

Network Theory

Comprehensive Theory

with **Solved Examples** and **Practice Questions**

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CHAPTER 7

Second Order RLC Circuits

7.1 Introduction

In the previous chapter we studied circuits which contained only one energy storage element, combined with a passive network which partly determined how long it took either the capacitor or the inductor to charge/discharge. The differential equations which resulted from analysis were always first-order. In this chapter we will consider circuits containing two storage elements. These are known as second-order circuits because their responses are described by differential equations that contain second derivatives.

Our analysis of second-order circuits will be similar to that used for first-order. We will first consider circuits that are excited by the initial conditions of the storage elements. Although these circuits may contain dependent sources, they are free of independent sources. These source-free circuits, will give natural responses as expected. Later we will consider circuits that are excited by independent sources. These circuits will give both the transient response and the steady-state response.

7.2 Finding Initial and Final Values

A circuit is given and we need to find the initial and final conditions $i(0^+)$, $v(0^+)$, $\frac{dv(0^+)}{dt}, \frac{di(0^+)}{dt}, i(\infty)$ and $v(\infty)$ here $i(t)$ and $v(t)$ are the inductor current and capacitor voltage respectively.

To determine $i(0^+)$, $v(0^+)$ keep in mind that the capacitor voltage is always continuous so that

$$v(0^+) = v(0^-) \quad \text{...(7.1)}$$

and the inductor current is always continuous so that

$$i(0^+) = i(0^-) \quad \text{...(7.2)}$$

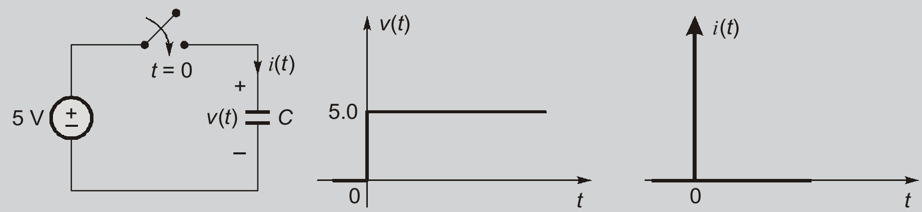
where $t = 0^-$ denotes the time just before a switching event and $t = 0^+$ is the time just after the switching event, assuming that the switching event takes place at $t = 0$.

To determine $\frac{dv(0^+)}{dt}, \frac{di(0^+)}{dt}$ draw the circuit at $t = 0^+$ (the circuit obtained after switching) and then try to write the KVL or KCL equation, employing either the nodal equation or loop equation which give you required quantity more directly, then simply put condition $t = 0^+$, and find the required quantity.

To determine $i(\infty)$, $v(\infty)$ that is value of current and voltage at steady state (since at steady state the capacitor is open circuit and inductor is short circuit) so $i(\infty)$, $v(\infty)$ are the short circuit inductor current and open circuit capacitor voltage respectively. The examples at the end of the chapter illustrate these ideas.

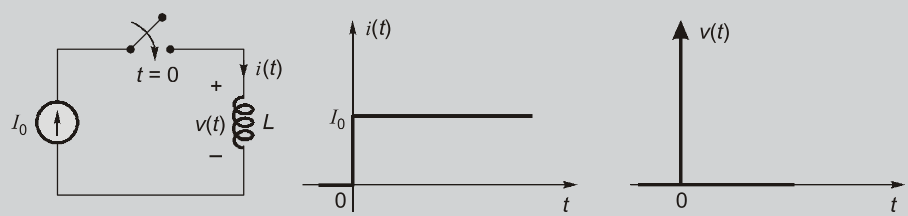
NOTE

- Till now we have considered that the voltage across capacitor will not change abruptly i.e. $v(0^-) = v(0^+)$ but there is an exception case as shown in the figure.



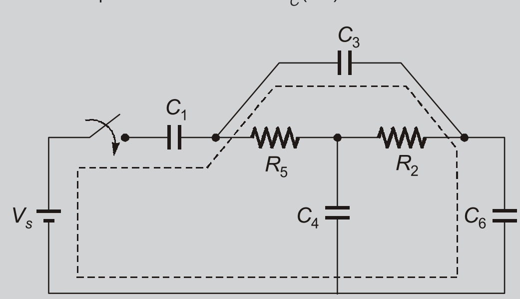
Initially the capacitor was not charged and $v(0^-) = 0$. Now the battery is connected to the circuit at $t = 0$, since the battery is ideal with $R_{Th} = 0$, so time constant of circuit $\tau = RC = 0$. So capacitor charges to 5 V instantaneously i.e. $v(0^-) \neq v(0^+)$ and current will be an impulse function in this case.

- Till now we have considered that the current through inductor will not change abruptly i.e. $i(0^-) = i(0^+)$ but there is an exception case as shown in the figure.



Initially the inductor was not charged and $i(0^-) = 0$. Now the current source is connected to the circuit at $t = 0$, since the current source is ideal with $R_{Th} = \infty$, so time constant of circuit $\tau = L/R = 0$. So inductor charges to I_0 instantaneously i.e. $i(0^-) \neq i(0^+)$ and voltage will be an impulse function in this case.

- Consider the circuit given in the figure below, the switch is turn on at $t = 0$ and initial charge on each capacitor is 0 i.e. $v_c(0^-) = 0$.



All capacitors are initially uncharged and so circuit at $t = 0^+$, will have all the capacitors acting as short-circuit if we consider the path shown in the above circuit we can see that the current at $t = 0^+$ will be ∞. Thus, $v_c(0^+) \neq v_c(0^-)$ for C_1, C_6, C_3 and $v_c(0^+)$ for these capacitors can be found using voltage division rule.

7.3 The Source-Free Series *RLC* Circuit

Consider the series *RLC* circuit shown in Figure (7.1). The circuit is being excited by the energy initially stored in the capacitor and inductor. The energy is represented by the initial capacitor voltage V_0 and initial inductor current I_0. Thus, at $t = 0$,

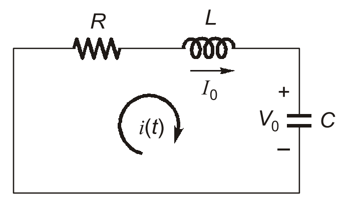


***Figure-7.1 :** A source-free series RLC circuit*

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i(t)\,dt = V_0 \qquad \text{...(7.3)}$$

$$i(0) = I_0 \qquad \text{...(7.4)}$$

Applying KVL around the loop in Figure (7.1)

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt = 0 \qquad \text{...(7.5)}$$

With the help of Laplace transform

$$i(t) \xrightarrow{\mathcal{L}} I(s)$$

$$\frac{di(t)}{dt} \xrightarrow{\mathcal{L}} sI(s) - i(0^-)$$

$$\int_{-\infty}^{t} i(t)\,dt \xrightarrow{\mathcal{L}} \frac{I(s)}{s} + \frac{\int_{-\infty}^{0} i(t)\,dt}{s}$$

The equation (7.5) changes to, $RI(s) + L[sI(s) - I_0] + \frac{1}{C}\left[\frac{I(s)}{s} + \frac{V_0}{s}\right] = 0$

$$I(s) = \left\{\frac{I_0 s - \dfrac{V_0}{LC}}{s^2 + s\dfrac{R}{L} + \dfrac{1}{LC}}\right\} \qquad \text{...(7.6)}$$

The denominator of the above equation is known as characteristic equation of the circuit. Roots of denominator of the above equation are

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad \text{...(7.7)}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad \text{...(7.8)}$$

A more compact way of expressing the roots is

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \quad \text{...(7.9)}$$

$$\alpha = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

The roots s_1 and s_2 are called natural frequencies, measured in nepers per second (Np/s), because they are associated with the natural response of the circuit; ω_0 is known as the resonant frequency or strictly as the undamped natural frequency, expressed in radians per second (rad/s); and α is the neper frequency or the damping factor, expressed in nepers per second. In terms of α and ω_0, the denominator of the equation (7.6) can be written as

$$s^2 + 2\alpha s + \omega_0^2 = 0 \quad \text{...(7.10)}$$

The solution of equation (7.10) depends on α and ω_0. The ratio $\frac{\alpha}{\omega_0}$ is known as the damping ratio ξ. Thus, the natural response of series *RLC* circuit is

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where the constants A_1 and A_2 are determined from the initial values $i(0)$ and $\frac{di(0)}{dt}$.

From equation (7.9), we can infer that there are three types of solutions:

1. If $\alpha > \omega_0$, both the roots will be real and unequal, the response is said to be over-damped.
2. If $\alpha = \omega_0$, both the roots are real and equal, the response is said to be critically damped.
3. If $\alpha < \omega_0$, both the roots will be complex and conjugates of each other, the response is said to be under-damped.

We will consider each of these cases separately.

Overdamped Case ($\alpha > \omega_0$)

From equations (7.8) and (7.9), $\alpha > \omega_0$ implies $C > 4L/R^2$. When this happens, both roots s_1 and s_2 are negative and real. The response is

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Where A_1 and A_2 are found from the initial conditions. This shows that the natural response is the sum of two decaying exponential. For the over-damped response, damping ratio $\xi > 1$. Figure (7.2) illustrates a typical overdamped case.

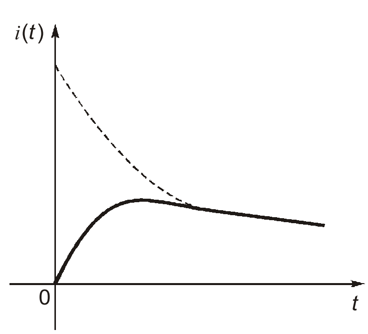
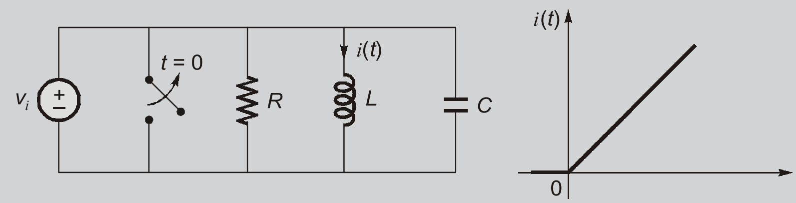


***Figure-7.2 :** Overdamped response*

- We considered the step response of the parallel *RLC* circuit where the input was current and response was the voltage across inductor. If we consider the response of parallel *RLC* circuit with voltage as input and current across the inductor as output then current across the inductor $i(t)$ will rise linearly as shown in the figure. The circuit in this case is an unstable circuit.



7.8 Circuit Analysis in the s-Domain

All the circuit analysis techniques that we have studied for pure resistive networks may be used in *s*-domain analysis. The node voltage method, mesh current method, source transformations, and Thevenin-Norton equivalents are all valid techniques in the *s*-domain. These can be applied using same methodologies as we discussed for resistive networks. The step-by-step procedure of circuit analysis in the *s*-domain is given below.

- **Step-1 :** Draw the circuit into *s*-domain substituting an *s*-domain equivalent for each circuit element. The inductors and capacitors are replaced by their equivalent discussed in previous chapter (For inductor and capacitor we first need to determine the initial capacitor voltage and inductor current and then replace them with their *s*-domain equivalent).
- **Step-2 :** Apply any circuit analysis technique to obtain the desired voltage or current in the *s*-domain.
- **Step-3 :** Take inverse Laplace transform to convert the voltages and/or currents back to the time domain.

Using Laplace Transform to find which element get charged at steady-state

A circuit with inductor and capacitor is given and we need to determine that which element get charged at steady-state. For this we follow these steps:

- Convert the circuit from time domain into s-domain.
- Apply nodal analysis at the node connecting the inductor and capacitor.
- Now apply final value theorem to find value of voltage at node at $t = \infty$ and find which element get charged.

Now we shall illustrate these steps with the help of two examples.

Example-1: Consider the circuit given in the Figure 7.8, our motive is to determine which element get charged at steady-state.

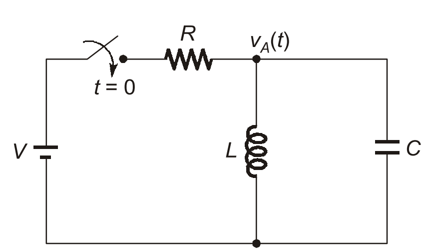
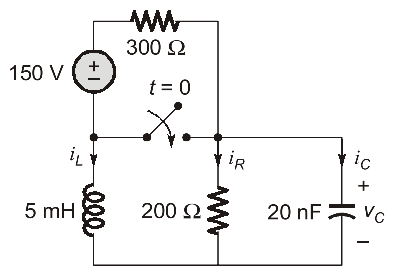


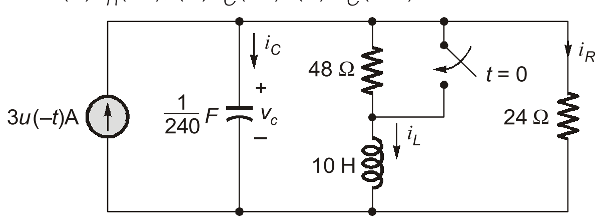
***Figure*-7.8**

Student's Assignments 1

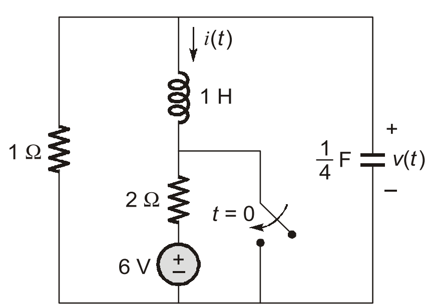
Q.1 Find an expression for $v_C(t)$ valid for $t > 0$ in the circuit of figure.



Q.2 After being open for a long time, the switch in figure, closes at $t = 0$. Find (a) $i_L(0^-)$; (b) $v_C(0^-)$; (c) $i_R(0^+)$; (d) $i_C(0^+)$; (e) $v_C(0.2)$.



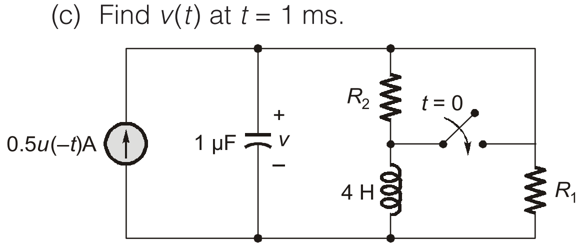
Q.3 Given the circuit in figure, find $i(t)$ and $v(t)$ for $t > 0$.



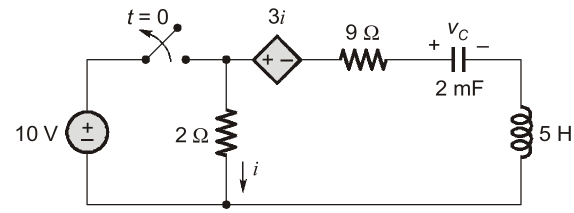
Q.4 (a) Choose R_1 in the circuit of figure, so that the response after $t = 0$ will be critically damped.

(b) Now find R_2 to obtain $v(0) = 100$ V.

(c) Find $v(t)$ at $t = 1$ ms.

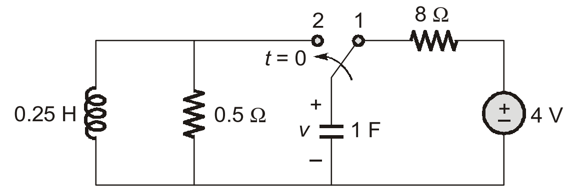


Q.5 Find an expression for $v_C(t)$ in the circuit of figure, valid for $t > 0$.

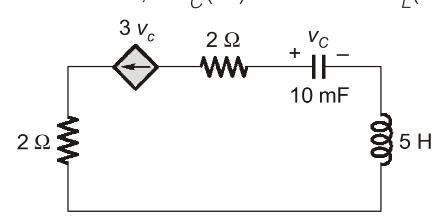


Q.6 In the circuit of figure, the switch has been in position 1 for a long time but moved to position 2 at $t = 0$. find

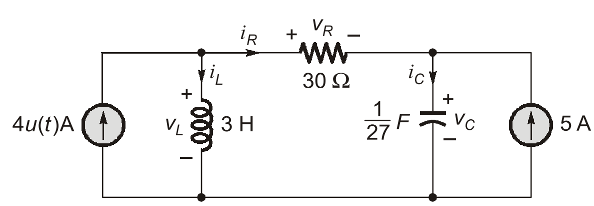
(a) $v(0^+)$, $dv(0^+)/dt$ (b) $v(t)$ and $t \geq 0$.



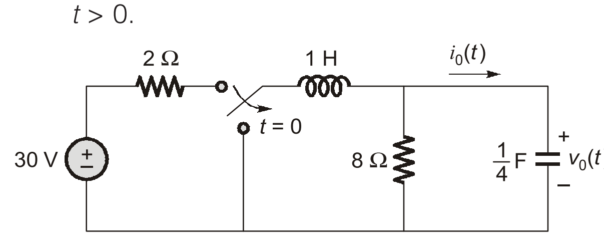
Q.7 Find an expression for $i_L(t)$ in the circuit of figure, valid for $t > 0$, if $v_C(0^-) = 10$ V and $i_L(0^-) = 0$.



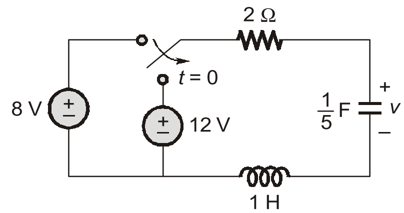
Q.8 Complete the determination of the initial conditions in the circuit of figure and determine the value of first derivative of all three voltage and current variable at $t = 0^+$.



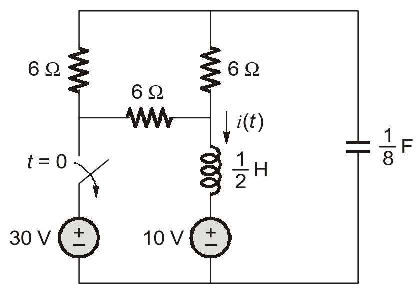
Q.9 In the circuit of figure, calculate $i_0(t)$ and $v_0(t)$ for $t > 0$.



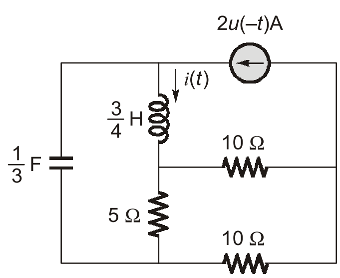
Q.10 Determine $v(t)$ for $t > 0$ in the circuit of figure.



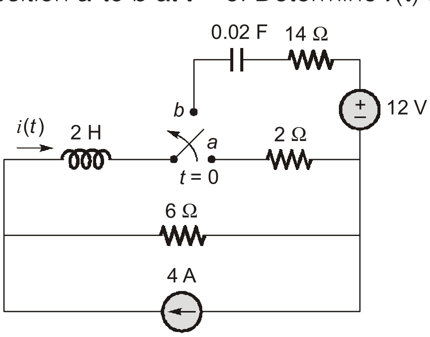
Q.11 For the network in figure, solve for $i(t)$ for $t > 0$.



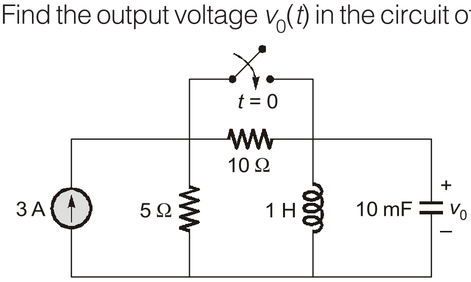
Q.12 Refer to the circuit in figure. Calculate $i(t)$ for $t > 0$.



Q.13 The switch in the circuit of figure is moved from position a to b at $t = 0$. Determine $i(t)$ for $t > 0$.



Q.14 Find the output voltage $v_0(t)$ in the circuit of figure.



Answer:

1. $(80e^{-50000t} - 20e^{-200000t}\ \text{V})$
2. (a) 1 A (b) 48 V (c) 2 A (d) –3 A (e) –17.54 V
3. $i(t) = (-2 - 2t)\,e^{-2t}$ A, $v(t) = (2 + 4t)\,e^{-2t}$ V
4. ($R_1 = 1$ kΩ, $R_2 = 250$ Ω, $v(1\text{ ms}) = -212$ V)
5. $-e^{-0.8t}(5\cos 9.89t + 0.413\sin 9.89t)$ V
6. (a) 4 V, –8 V/s (b) $e^{-t}(4\cos 1.73t - 2.30\sin 1.73t)$ V
7. $(-30e^{-300t}\ \text{A})$
8. $V_L(0^+) = 120$ V, $V_C(0^+) = 150$ V, $V_R(0^+) = -30$ V

$$\frac{dV_L(0^+)}{dt} = -1092\text{ V/s},\quad \frac{dV_C(0^+)}{dt} = 108\text{ V/s},$$

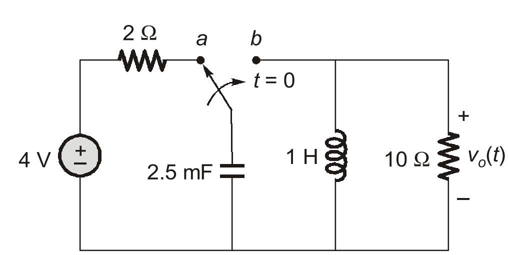
$$\frac{dV_R(0^+)}{dt} = -1200\text{ V/s},\quad \frac{di_R(0^+)}{dt} = -40\text{ A/sec}$$

$$\frac{di_c(0^+)}{dt} = -40\text{ A/sec}$$

9. $v_0(t) = e^{-t/4}(24\cos 1.98t + 3.02\sin 1.98t)$ V
   $i_0(t) = e^{-t/4}(0.00013\cos 1.98t - 12.09\sin 1.98t)$ A
10. $12 - e^{-t}(4\cos 2t + 2\sin 2t)$ V
11. $(5\,e^{-4t})$ A **12.** $(e^{-4.4t} + e^{-0.903t})$ A
13. $(3 - 9t)\,e^{-5t}$ A **14.** $(200t\,e^{-10t})$ V

Student's Assignments 2

Q.1 The switch in the circuit of figure below has been in position 'a' for a long time. At $t = 0$, the switch moves instantaneously to position 'b'.

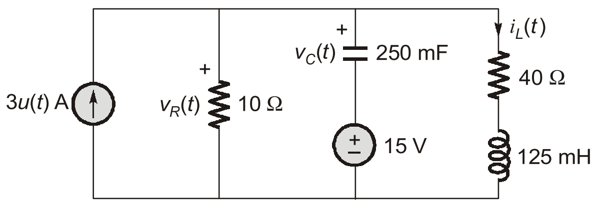


If $v(t) = (Ae^{-20t} - Bte^{-20t})$ V, for $t > 0$, then the values of A and B respectively are

(a) 0, 80 (b) 4, 240
(c) 4, 80 (d) 4, 0

Common Data Questions (9 to 10):

Consider the circuit shown in the figure, where $u(t)$ is given as unit step function.



Q.9 What are the values of $v_C(t)$ and $v_R(t)$, at $t = 0^+$ respectively?

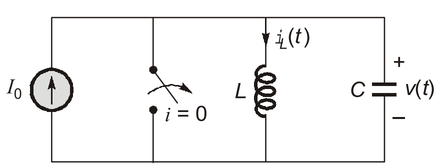
(a) 0, 0 (b) –15 V, –30 V
(c) 0, 30 V (d) –15 V, 0

Q.10 Which of the following sets of values is correct for the circuit at $t = 0^+$?

	$\frac{di_L(0^+)}{dt}$	$\frac{dv_C(0^+)}{dt}$	$\frac{dv_R(0^+)}{dt}$
(a)	30 V/s	12 V/s	0
(b)	0	12 V/s	0
(c)	0	0	0
(d)	0	12 V/s	12 V/s

Q.11 The switch in the figure has been closed for a long time and is opened at $t = 0$. The inductor current and the capacitor voltage are zero at $t = 0$. For $t \geq 0$, $v(t)$ is given by

(Given $\omega_0 = \frac{1}{\sqrt{LC}}$)



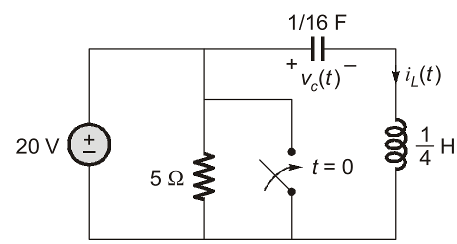
(a) $v(t) = \left(\frac{I_0}{\omega_0 C}\right)\sin\omega_0 t$

(b) $v(t) = (I_0\omega_0 L)\cos\omega_0 t$

(c) $v(t) = \left(\frac{I_0}{\omega_0 L}\right)\sin\omega_0 t$

(d) $v(t) = (I_0\omega_0 C)\cos\omega_0 t$

Q.12 In the circuit shown below the switch is closed at $t = 0$ after long time. The current $i_L(t)$ for $t > 0$ is



(a) $-10\sin 8t$ A (b) $10\sin 8t$ A
(c) $-10\cos 8t$ A (d) $10\cos 8t$ A

Answer Key:

1. (c)	**2.** (c)	**3.** (a)	**4.** (b)	**5.** (c)
6. (a)	**7.** (d)	**8.** (b)	**9.** (d)	**10.** (d)
11. (a)	**12.** (a)			