Electronics Engineering

Microwave Engineering

Comprehensive Theory

with **Solved Examples** and **Practice Questions**

MADE EASY Publications

MADE EASY Publications

Corporate Office: 44-A/4, Kalu Sarai (Near Hauz Khas Metro Station), New Delhi-110016
E-mail: infomep@madeeasy.in
Contact: 011-45124660, 8860378007
Visit us at: www.madeeasypublications.org

Microwave Engineering

First Edition: 2015
Second Edition: 2016
Third Edition: 2017
Fourth Edition: 2018

Contents

Microwave Engineering

■■■

Microwave Engineering

Microwave Engineering

After studying the basic electromagnetic field theory and its application in understanding waves and transmission lines, we are now ready to understand the guided wave propagation through waveguides. We shall start with the fields inside waveguide and its parameters like cut-off frequency, guide wavelength, phase velocity and dominant mode of wave propagation etc. Circular waveguides and other special purpose waveguides are also discussed. Further, we shall proceed to the microwave components which are sections of waveguides and their analysis using *S*-parameters followed by ferrite devices and their applications.

The next part of the book deals with microwave sources (amplifiers and oscillators) both high power (tubes) and low power (solid state devices). The operation of tubes like Klystrons, magnetron, and TWTs are discussed exhaustively followed by solid state devices like tunnel diode, IMPATT diodes, Gunn diodes and MASERs etc. The last part of the book discusses microwave measurements in detail. The major emphasis throughout the book is to developing a reader to understand and analyze principles of operation of various microwave devices and circuits, which are always an integral part of various competitive examinations. Throughout this book, a sequential and comprehensive approach has been used, so that a beginner with EMT basics can utilize this book in an efficient manner.

Chapter 1

Introduction

1.1 Microwave Frequencies

The term ***microwaves*** is used to describe electromagnetic waves with frequencies ranging 300 MHz to 300 GHz. The corresponding wavelength $\left(\lambda = \dfrac{c}{f}\right)$ range is 1 m up to 1 mm for 300 GHz waves. Figure (1.1) shows the location of microwave frequency band in the electromagnetic spectrum.

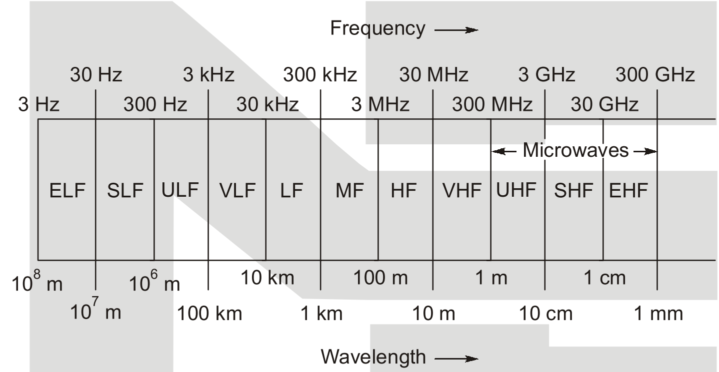

Figure-1.1

ELF	Extremely Low Frequency	3-30 Hz
SLF	Super Low Frequency	30-300 Hz
ULF	Ulta Low Frequency	300 Hz-3 kHz
VLF	Very Low Frequency	3 kHz- 30 kHz
LF	Low Frequency	30 kHz-300 kHz
MF	Medium Frequency	300 kHz-3 MHz
HF	High Frequency	3 MHz-30 MHz
VHF	Very High Frequency	30 MHz-300 MHz
UHF	Ultra High Frequency (decimeter waves)	300 MHz-3 GHz
SHF	Super High Frequency (centimeter waves)	3 GHz-30 GHz
EHF	Extremely High Frequency (millimeter waves)	30 GHz-300 GHz
	submillimeter waves	300 GHz-3000 GHz
IR	Infrared	3000 GHz-416,000 GHz

Table-1.1

1.2 Microwave Engineering

The term ***Microwave Engineering*** refers to the engineering and design of communication/ navigation systems in the microwave frequency range. The characteristic feature of microwave engineering is the ***short wavelengths*** involved, these being of the same order of magnitude as the circuit elements and devices employed.

The short wavelengths involved in turn means that the propagation time for electrical effects from one point in a circuit to another point is comparable with the period of oscillating currents and charges in the system. As a result, conventional low-frequency circuit analysis based on Kirchhoff's laws and voltage-current concepts no longer suffices for an adequate description of the electrical phenomena taking place. It is necessary instead to carry out the analysis in terms of the electric and magnetic field associated with the device. In essence, microwave engineering is applied electromagnetic fields engineering.

1.3 Microwave Properties and Advantages

1.3.1 High Bandwidth

At high frequencies more bandwidth (information-carrying capacity) can be realized. A 1% bandwidth at 600 MHz is 6 MHz (the bandwidth of a single TV channel), and at 60 GHz a 1% bandwidth is 600 MHz (100 TV channels).

1.3.2 Antenna Gain and Directivity

The gain of the antenna is directly proportional to its electrical size. The beamwidth of an antenna is inversely proportional to the electrical size of its maximum dimension. Thus shorter wavelengths at microwave frequencies allow for smaller antennas.

1.3.3 Effect of Ionosphere

When lower frequency waves are directed upward into the atmosphere, they experience significant reflection due to ionosphere. The higher frequency waves which pass through the ionosphere with little effect and are therefore utilized in satellite communications and space transmissions.

1.3.4 Line of Sight Transmission/ Reception

The microwave receiving antenna must be within the line-of-sight of the transmitting antenna. Long distance communication on earth requires that microwave relay stations be used.

1.3.5 Electromagnetic Noise Characteristics

The EM noise level in nature over the 1-10 GHz frequency is small. This allows for the detection of low signal levels using sensitive receivers.

1.3.6 Target Reflection of Electromagnetic Waves (Radar Cross Section)

In general, electrically large conducting radar targets reflect more energy (shape is also a factor - stealth design). Thus, the higher frequencies of microwaves are preferred for radar systems.

1.3.7 Power Requirements

At microwave frequencies, the power requirements of the transmitter become very small as compared to that at MF/HF, due to high gain of the antennas at microwaves.

1.3.8 Absorption of Resonant Frequencies

Various materials absorb microwave energy (dissipated in the form of heat) at specific resonant frequencies.

1.4 Applications of Microwaves

1.4.1 Wireless Communications

- Personal Communications Systems (PCS), (like pagers, cell phones etc.)
- Global Positioning Satellite (GPS) Systems
- Wireless Local Area Networks (WLANS)
- Direct Broadcast Satellite (DBS) Television
- Telephone Microwave/ Satellite Links, etc

1.4.2 Remote Sensing

Radar (acting remote sensing - radiate and receive)

- Military Applications (target tracking)
- Weather Radar
- Ground Penetrating Radar (GPR)
- Agricultural Applications

Radiometry (passive remote sensing - receive inherent emissions)

- Radio Astronomy

1.4.3 Industrial and Home Applications

- Cooking, drying, heating
- Microwave Spectroscopy: Molecular properties of materials can be determined by passing microwaves through a sample of material and measuring the absorption spectrum.

1.4.4 Medical Applications

- Microwave hyperthermia for cancer therapy: A therapy using non ionizing microwave radiation.
- Diagnostic with bioimpedances (BIA): Analysis of resistance and reactance in the human body.

Example - 1.1 Match List-I (IEEE Bands) with List-II (Frequency Range) and select the correct answer using the codes given below the lists:

List-I	List-II
A. C	1. 27 – 40 GHz
B. S	2. 4 – 8 GHz
C. Ka	3. 2 – 4 GHz
D. Ku	4. 12 – 18 GHz

Codes:

	A	B	C	D
(a)	1	3	2	4
(b)	1	4	2	3
(c)	2	3	1	4
(d)	2	4	1	3

Solution : (c)

Designation	Frequency range (in GHz)
HF	0.003 - 0.030
VHF	0.030 - 0.300
UHF	0.300 - 1.000
L band	1 - 2
S band	2 - 4
C band	4 - 8
X band	8 - 12
Ku band	12 - 18
K band	18 - 27
Ka band	27 - 40
Millimeter	40 - 300
Submillimeter	> 300

Example - 1.2 Which of the following frequency band of carriers can be used to obtain bandwidth of the order of 10 MHz.

(a) HF (b) VHF
(c) Microwaves (d) UHF

Solution : (c)

Microwave frequencies range from 300 MHz to 300 GHz. Therefore, bandwidth of the order of 10 MHz can be realized easily using carriers in microwave range.

Example - 1.3 Microwave frequencies are used for satellite communications because they do not suffer

(a) Refraction by ionosphere (b) Attenuation in space
(c) Phase distortion (d) Fading

Solution : (a)

Microwaves penetrate into ionosphere.

Example - 1.4 Consider the following statements:

1. High bandwidth
2. Smaller antennas
3. LOS communication

Use of microwaves in system application results in which of these advantages?

(a) 1 and 2 (b) 2 and 3
(c) 3 and 1 (d) 1, 2 and 3

Solution : (d)

$$f\uparrow \Rightarrow \text{Bandwidth}\uparrow$$
$$f\uparrow \Rightarrow \lambda\downarrow$$
$$\Rightarrow \text{Size of component}\downarrow$$

CHAPTER 2

Waveguides

2.1 Waveguides

Waveguides, like transmission lines, are structures used to guide Electromagnetic waves from one point (source) to another (load). Maxwell's equations predict that electromagnetic waves can also be guided through metallic tubes, like water is guided through pipes. Two common metallic waveguides, rectangular and circular cross section are shown in Figure 2.1

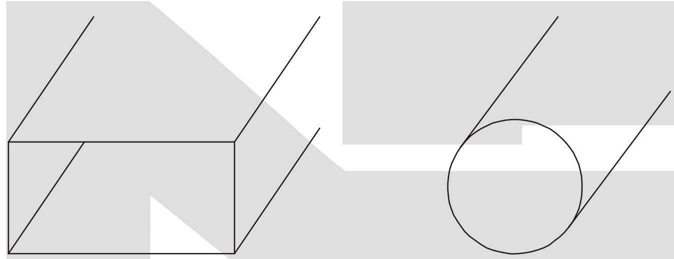

Figure-2.1

Comparison of Waveguide and Transmission Line Characteristics

Transmission Line	Waveguide
• Two or more conductors separated by some insulating medium (Two wire, microstrip coaxial etc)	• Metal waveguides are typically one enclosed conductor filled with an insulating medium (Rectangular, circular)
• Normal operating mode is the TEM or quasi-TEM mode (can support TE and TM modes but these modes are undesirable)	• Operating modes are TE or TM modes (cannot support a TEM mode)
• No cutoff frequency for the TEM mode. Transmission lines can transmit signals from DC upto high frequency.	• Must operate the waveguide at a frequency above the respective TE or TM mode cutoff frequency for that mode to propagate.
• Significant signal attenuation at high frequencies due to conductor and dielectric losses.	• Lower signal attenuation at high frequencies than transmission lines.
• Small cross section transmissions lines (like coaxial cables) can only transmit low power levels.	• Metal waveguides can transmit high power levels.
• Large cross section transmission lines (like power transmission lines) can transmit high power levels.	• Large cross section (low frequency) waveguides are impractical due to large size and high cost.

Example - 2.1 The main difference between the operation of transmission lines and waveguides is that

(a) the latter are not distributed, like transmission lines
(b) the former can use stubs and quarter-wave transformers, unlike the latter
(c) terms such as impedance matching and SWR cannot be applied to waveguides
(d) Transmission lines use the principal mode of propagation and therefore do not suffer from low-frequency cut-off

Solution : (d)

Principal mode is TEM mode and this mode has zero cutoff frequency. Waveguides allows signals after particular range only.

Example - 2.2 Match List-I (Types of transmission line structures) with List-II (Modes of propagation) and select the correct answer using the code given below the lists:

List-I	List-II
A. Coaxial	1. Quasi TEM
B. Hollow rectangular waveguide	2. Pure TEM
C. Microstrip	3. TE/TM
D. Hollow cylindrical waveguide	4. Hybrid

Codes:

	A	B	C	D
(a)	2	1	3	4
(b)	4	3	1	3
(c)	2	3	1	3
(d)	2	3	1	2

Solution : (c)

Example - 2.3 Which of the following transmission line structures allow TEM mode of wave propagation?

(a) Two wire transmission line
(b) Coaxial cable
(c) Rectangular waveguide
(d) Both (a) and (b)

Solution : (d)

Example - 2.4 Statement (I): Coaxial cable is not preferred at microwave frequencies.

Statement (II): At microwave frequencies, coaxial cable has high attenuation.

(a) Both Statement (I) and Statement (II) are individually true and Statement (II) is the correct explanation of Statement (I)
(b) Both Statement (I) and Statement (II) are individually true but Statement (II) is not the correct explanation of Statement (I)
(c) Statement (I) is true but Statement (II) is false
(d) Statement (I) is false but Statement (II) is true

Solution : (a)

At high frequencies (microwave frequencies) attenuation in coaxial cable is very high.

2.1.1 General Wave Equations in Rectangular Coordinates

Given any time-harmonic source of electromagnetic radiation, the phasor electric and magnetic fields associated with the electromagnetic waves that propagate away from the source through a medium characterized by ($\mu, \in$) must satisfy the source free Maxwell's equations (in phasor form) given by

$$\nabla \times \bar{E} = -j\omega\mu\bar{H} \tag{2.1}$$

$$\nabla \times \bar{H} = j\omega \in \bar{E} \tag{2.2}$$

$$\nabla \cdot \bar{E} = 0 \tag{2.3}$$

$$\nabla \cdot \bar{H} = 0 \tag{2.4}$$

Now, we manipulate source-free Maxwell's equations to get wave equations for electric and magnetic fields.

Taking the curl of equation (2.1) $\quad \nabla \times \nabla \times \bar{E} = -j\omega\mu\left(\nabla \times \bar{H}\right)$

and inserting equation (2.2) gives $\quad \nabla \times \nabla \times \bar{E} = (-j\omega\mu)(j\omega \in)\bar{E}$ (2.5)

Using the vector identity $\quad \nabla \times \nabla \times \bar{F} = \nabla\left(\nabla \cdot \bar{F}\right) - \nabla^2 \bar{F}$ $\quad$ (for any vector $\bar{F}$)

in equation (2.5) gives $\quad \nabla\left(\nabla \cdot \bar{E}\right) - \nabla^2 \bar{E} = \omega^2 \mu \in \bar{E}$ (2.6)

From equation (2.3), we see that the divergence of electric field is zero in a source free region i.e., $\nabla \cdot \bar{E} = 0$

Inserting this result into equation (2.6) gives, $\nabla^2 \bar{E} + \omega^2 \mu \in \bar{E} = 0$ (2.7a)

Similarly for magnetic field, $\nabla^2 \bar{H} + \omega^2 \mu \in \bar{H} = 0$ (2.7b)

If we let $\boxed{-\omega^2 \mu \in = \gamma^2}$

We find that the electric and magnetic field phasors satisfy vector wave equations given by

$$\boxed{\nabla^2 \bar{E} - \gamma^2 \bar{E} = 0} \tag{2.8a}$$

$$\boxed{\nabla^2 \bar{H} - \gamma^2 H = 0} \tag{2.8b}$$

where γ is intrinsic propagation constant of medium generally given by

$\boxed{\gamma = \alpha + j\beta}$ $\quad$ α-attenuation constant (Nep/m)

β-phase constant (rad/m)

For source free and loss less medium $\boxed{\gamma = j\beta = j\omega\sqrt{\mu \in}}$

Example - 2.5 For lossless propagation of EM waves through an unbounded dielectric, the propagation constant is

(a) real
(b) imaginary
(c) complex
(d) None of these

Solution : (b)

For lossless transmission $\alpha = 0$.

2.2 Rectangular Waveguides

Consider the geometry of a rectangular waveguide shown in Fig. 2.2, where *a* and *b* are the inner dimensions of the waveguide.

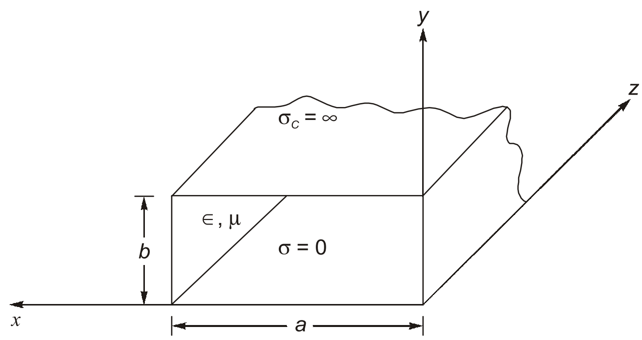

Figure-2.2

Assumptions:

- The waveguide is infinitely long, oriented along *z*-axis, and uniform along its length.
- The waveguide is filled with a source free ($\rho_v = 0$) lossless dielectric ($\in, \mu, \sigma = 0$) and guide walls are perfectly conducting ($\sigma_C = \infty$).
- Fields are time harmonic i.e. $e^{j\omega t}$ dependence.

The electric and magnetic fields of a general wave propagating in +*z*-direction through waveguide with guide propagation constant γ_g are characterised by a *z*-dependence of $e^{-\gamma_g z}$. Then general field (*E* & *H*) expressions inside the guide are given by

$$\bar{E}(x,y,z) = \bar{E}(x,y)e^{-\gamma_g z} = \left[E_x(x,y)\hat{x} + E_y(x,y)\hat{y} + E_z(x,y)\hat{z}\right]e^{-\gamma_g z} \quad (2.9a)$$

$$\bar{H}(x,y,z) = \bar{H}(x,y)e^{-\gamma_g z} = \left[H_x(x,y)\hat{x} + H_y(x,y)\hat{y} + H_z(x,y)\hat{z}\right]e^{-\gamma_g z} \quad (2.9b)$$

where $\boxed{\gamma_g = \alpha_g + j\beta_g}$

α_g = waveguide attenuation constant (*Nep*/*m*)

β_g = waveguide phase constant (rad/*m*)

The propagation constant is purely imaginary $(\alpha_g = 0, \gamma_g = j\beta_g)$ when the wave travels without attenuation (no losses) or complex-valued when losses are present.

Task

To obtain complete *E* and *H* field of a wave propagating in +*z* direction inside a rectangular waveguide of infinite length.

Method of Solution

1. Express transverse field components (E_x, E_y, H_x, H_y) in terms of Axial field components (E_z, H_z).
2. Obtain solution for the Axial/ Longitudinal fields (E_z, H_z) from the wave equations.
3. Obtain E_x, E_y, H_x, H_y from E_z and H_z.

Transverse components from Longitudinal/ Axial components

(E_x, E_y, H_x, H_y) $\qquad$ (E_z, H_z)

From equation (2.9), note that the derivatives of the transverse field components with respect to z are

$$\left.\begin{aligned} \frac{\partial E_x}{\partial z} = -\gamma_g E_x \qquad \frac{\partial E_x}{\partial z} = -\gamma_g E_y \\ \frac{\partial H_x}{\partial z} = -\gamma_g H_x \qquad \frac{\partial H_x}{\partial z} = -\gamma_g H_y \end{aligned}\right\} \tag{2.10}$$

Expanding the curl operator of the source free Maxwell's equations in rectangular coordinates

$$\nabla \times \bar{E} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{bmatrix} = -j\omega\mu\left[H_x\hat{x} + H_x\hat{y} + H_x\hat{z}\right] \tag{2.11a}$$

$$\nabla \times \bar{H} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{bmatrix} = j\omega \in \left[E_x\hat{x} + E_x\hat{y} + E_x\hat{z}\right] \tag{2.11b}$$

Equating vector components on both sides of (2.11) and using (2.10), we get

$$\frac{\partial E_z}{\partial y} + \gamma_g E_y = -j\omega\mu H_x \tag{2.12a}$$

$$-\gamma_g E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \tag{2.12b}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \tag{2.12c}$$

$$\frac{\partial H_z}{\partial_y} + \gamma_g H_y = j\omega\in E_x \tag{2.13a}$$

$$-\gamma_g H_x - \frac{\partial H_z}{\partial x} = -j\omega\mu E_y \tag{2.13b}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_z}{\partial y} = -j\omega\mu\in E_z \tag{2.13c}$$

The above six equations can be solved for the four transverse field components (E_x, E_y, H_x, H_y) in terms of E_z and H_z.

Eliminate H_y from (2.12b) and (13a) to obtain

$$E_x = \frac{-\gamma_g}{k_c^2}\frac{\partial E_z}{\partial x} - \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y} \tag{2.14a}$$

Eliminate H_x from equation (2.12a) and equation (2.13b) to obtain

$$E_y = \frac{-\gamma_g}{k_c^2}\frac{\partial E_z}{\partial y} + j\omega\mu\frac{\partial H_z}{\partial x} \tag{2.14b}$$

Finally put equation (2.14b) into equation (2.13b) and equation (2.14a) into equation (2.13a) to obtain

$$H_x = \frac{j\omega\in}{k_c^2}\frac{\partial E_z}{\partial y} - \frac{\gamma_g}{k_c^2}\frac{\partial H_z}{\partial x} \quad (2.14c)$$

$$H_y = \frac{-j\omega\in}{k_c^2}\frac{\partial E_z}{\partial x} - \frac{\gamma_g}{k_c^2}\frac{\partial H_z}{\partial y} \quad (2.14d)$$

where $$k_c^2 = \omega^2\mu\in + \gamma_g^2 = \gamma_g^2 - \gamma^2 \quad (2.15)$$

k_c : Cutoff wave number (rad/m)

γ_g : Guide propagation constant (defined for bounded medium)

γ : Intrinsic propagation constant (defined for unbounded medium)

The equations for transverse fields [equation (2.14)] in terms of longitudinal/axial fields (E_z, H_z) describe the different types of possible waveguide modes.

Wave Mode Classification

It is convenient to classify waveguide modes as to whether E_z or H_z exists.

Mode		
TEM	$E_z = 0$	$H_z = 0$
TE	$E_z = 0$	$H_z \neq 0$
TM	$E_z \neq 0$	$H_z = 0$
Hybrid (HE)	$E_z \neq 0$	$H_z \neq 0$

TEM Mode: ($E_z = 0$, $H_z = 0$)

In the Transverse Electromagnetic mode, both E and H fields are transverse to the direction of wave propagation. From equation (2.14) all field components (E_x, E_y, H_x, H_y) vanish for $E_z = 0$, $H_z = 0$. Thus we can conclude that rectangular waveguide cannot support TEM mode.

For TEM modes, the only way for the transverse fields to be non-zero with $E_z = H_z = 0$ is for $k_c = 0$, which yields [from equation (2.15)].

$$\gamma_g = \gamma = j\beta = \alpha_g + j\beta_g \Rightarrow \boxed{\beta_g = \beta}\text{ [TEM modes]} \quad (2.16)$$

Thus for unguided TEM waves (plane waves) moving through a lossless medium or guided TEM waves (waves on a transmission line) propagating on an ideal transmission line, we have $\gamma_g = j\beta = j\beta_g$.

Waveguide Modes: (TE, TM, Hybrid)

For the waveguide modes, k_c cannot be zero since this would yield unbounded results for transverse fields [(E_x, E_y, H_x, H_y of equation (2.14)]. Thus $\beta_g \neq \beta$ for waveguides and waveguide propagation constant can be written as

$$\gamma_g = \sqrt{k_c^2 - \omega^2\mu\in} \qquad \left[\gamma^2 = -\omega^2\mu\in \text{ for lossless dielectric}\right] \quad (2.17)$$

Case-I: (Cutoff)

If $$k_c^2 = \omega^2\mu\in$$

then $$\gamma_g = 0 \text{ or } \alpha_g = 0 = \beta_g$$

The value of ω that satisfies the above condition is called cutoff angular frequency given by

$$\boxed{\omega_c = \frac{1}{\sqrt{\mu \in}} k_c} \quad (2.18)$$

At this frequency no propagation takes place.

Case-II (Evanescent/ Attenuation)

If $\quad k_c^2 > \omega^2\mu \in \Rightarrow \omega_c^2\mu \in > \omega^2\mu \in \quad$ [from equation (2.18)]

then $\quad \boxed{\gamma_g = \alpha_g}, \quad \beta_g = 0$

In this case there will be no wave propagation because propagation constant is a real quantity. In other words *if the operating frequency is below the cutoff frequency, the wave will be attenuated*.

Case-III (Propagation)

If $\quad k_c^2 < \omega^2\mu \in \Rightarrow \omega_c^2\mu \in < \omega^2\mu \in$

then $\quad \boxed{\gamma_g = j\beta_g}, \quad \alpha_g = 0$

In this case there will be wave propagation as propagation constant is an imaginary quantity. In other words *if the operating frequency is above the cutoff frequency then wave propagation takes place*.

Summary of Wave Propagation Constant (γ_g)

If $f < f_c$, $\gamma_g = \alpha_g$ (real) $\quad e^{-\gamma_g z} = e^{-\alpha_g z} \quad$ Waves are attenuated (Evanescent modes)

If $f > f_c$, $\gamma_g = j\beta_g$ (imaginary) $\quad e^{-\gamma_g z} = e^{-j\beta_g z} \quad$ Waves are Unattenuated (Propagating modes)

2.2.1 Rectangular Waveguide TM Modes

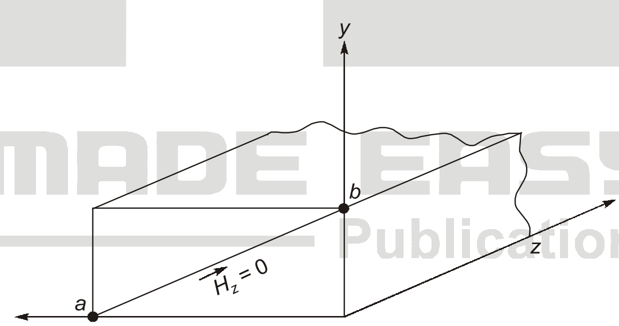

Figure-2.3

Transverse Magnetic (TM) waves in a rectangular guide are characterised by $H_z = 0$ and $E_z \neq 0$.

Transverse field equations (2.14) reduce to

$$E_x = \frac{-j\beta_g}{k_c^2}\frac{\partial E_z}{\partial x} \quad (2.19a)$$

$$E_y = \frac{-j\beta_g}{k_c^2}\frac{\partial E_z}{\partial y} \quad (2.19b)$$

$$H_x = \frac{j\omega \in}{k_c^2}\frac{\partial E_z}{\partial y} \quad (2.19c)$$

$$H_y = \frac{-j\omega \in}{k_c^2}\frac{\partial E_z}{\partial x} \tag{2.19d}$$

where $k_c \neq 0$ and $\beta_g^2 - \omega^2 \mu \in = -k_c^2$ is replaced.

To apply equation (2.19), one must find E_z from the Helmholtz wave equation

$$\nabla^2 E_z + \omega^2 \mu \in E_z = 0 \tag{2.20}$$

Which expanded in rectangular coordinates is

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} + \omega^2 \mu \in E_z = 0 \tag{2.21}$$

The electric field function may be determined using separation of variables by assuming a solution of the form

$$E_z = X(x)Y(y)e^{-j\beta_g z} \qquad (+\hat{z} \text{ travelling waves}) \tag{2.22}$$

Inserting equation (2.22) into equation (2.21) gives

$$Y\frac{d^2X}{dx^2}e^{-j\beta_g z} + X\frac{d^2Y}{dy^2}e^{-j\beta_g z} + \left(\omega^2\mu \in -\beta_g^2\right)e^{-j\beta_g z}XY = 0 \tag{2.23}$$

Dividing equation (2.23) by equation (2.22) gives

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + k_c^2 = 0 \tag{2.24a}$$

where $$\boxed{k_c^2 = \omega^2 \mu \in -\beta_g^2} \tag{2.24b}$$

NOTE

Note that first term in equation (2.24a) is a function x only and second term is a function of y only. In order for equation (2.24a) to be satisfied for every x and y within the waveguide, each of the first two terms in the equation (2.24a) must be constant (as third term is constant i.e. k_c^2)

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2$$

$$\Rightarrow \quad \frac{d^2X}{dx^2} + k_x^2 X = 0 \tag{2.25a}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2$$

$$\Rightarrow \quad \frac{d^2Y}{dy^2} + k_y^2 Y = 0 \tag{2.25b}$$

and $$\boxed{k_c^2 = k_x^2 + k_y^2} \quad \text{(Separation equation)} \tag{2.26}$$

The original second order partial differential equation (2.24a) dependent on two variables (x and y) has been separated into two second order ordinary differential equations (2.25) each dependent on only one variable.

The general solutions to the two separate differential equations (2.25) are

$$X(x) = A\sin k_x X + B\cos k_x X \quad (2.27a)$$

$$Y(y) = C\sin k_y Y + D\cos k_y Y \quad (2.27b)$$

The general solution for E_z can be written by substituting equation (2.27) into equation (2.22)

$$E_z(x, y, z) = \left(A\sin k_x X + B\cos_x X\right)\left(C\sin k_y X + D\cos_y Y\right)e^{-j\beta_g z} \quad (2.28)$$

To evaluate constants (A, B, C, D) in equation (2.28) we must apply boundary conditions. Boundary conditions say that the tangential components of E_z vanish on the walls of rectangular waveguide.

$$E_x(x, y) = 0 \quad \text{at} \quad x = 0, a \quad \text{(right and left walls)} \quad (2.29a)$$

$$E_y(x, y) = 0 \quad \text{at} \quad y = 0, b \quad \text{(bottom and top walls)} \quad (2.29b)$$

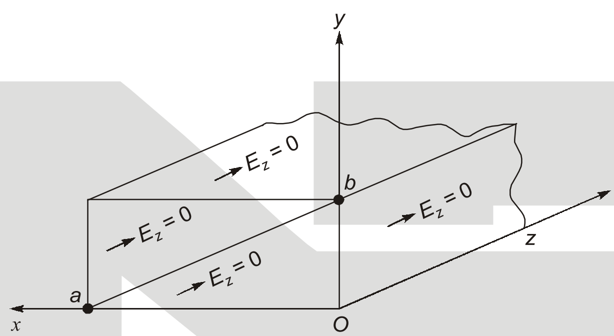

Figure-2.4

The application of boundary conditions yields,

$$E_z = 0 \text{ at } x = 0 \Rightarrow B = 0 \quad (2.30a)$$

$$E_z = 0 \text{ at } x = a \Rightarrow k_x a = m\pi\,(m = 1, 2, ...) \Rightarrow k_x = \frac{m\pi}{a} \quad (2.30b)$$

$$E_z = 0 \text{ at } y = 0 \Rightarrow D = 0 \quad (2.30c)$$

$$E_z = 0 \text{ at } y = b \Rightarrow k_y b = n\pi\,(n = 1, 2, ...) \Rightarrow k_y = \frac{n\pi}{b} \quad (2.30d)$$

Substituting equation (2.30) into equation (2.28) gives

$$\boxed{E_z(x, y, z) = E_0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-j\beta_g z}} \quad (2.31)$$

where E_0 is amplitude constant ($E_0 = A \times C$).

The number of discrete TM modes is infinite based on the possible values of m and n. An individual TM mode is designated as the TM_{mn} mode.

The transverse field components for the TM_{mn} mode can be computed from equation (2.19)

$$E_x = \frac{-j\beta_g}{k_c^2}\frac{\partial E_z}{\partial x} = \frac{-j\beta_g}{k_c^2}\left(\frac{m\pi}{a}\right)E_0\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-j\beta_g z} \quad (2.32a)$$

$$E_y = \frac{-j\beta_g}{k_c^2}\frac{\partial E_z}{\partial y} = \frac{-j\beta_g}{k_c^2}\left(\frac{n\pi}{b}\right)E_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{-j\beta_g z} \quad (2.32b)$$

$$H_x = \frac{j\omega \in}{k_c^2}\frac{\partial E_z}{\partial y} = \frac{j\omega \in}{k_c^2}\left(\frac{n\pi}{b}\right)E_0 \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{-j\beta_g z} \quad (2.32c)$$

$$H_y = -\frac{j\omega \in}{k_c^2}\frac{\partial E_z}{\partial x} = \frac{-j\omega \in}{k_c^2}\left(\frac{m\pi}{a}\right)E_0 \cos\frac{m\pi x}{a}\sin\frac{n\pi}{b}e^{-j\beta_g z} \quad (2.32d)$$

where $$\boxed{k_c^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$ [obtained from equation (2.26)] (2.33)

NOTE

- Integers m and n represents the number of half cycle variations of magnetic field in the x and y direction respectively.
- TM_{m0}, TM_{0n}, modes and TM_{00} mode doesn't exist as all field components [equations (2.31) and (2.32)] vanish.
- TM_{11} is the lowest order mode of all TM_{mn} modes.

Parameters of TM_{mn} Modes

1. Cut-off Wave number: (k_c)

From equation (2.33) and equation (2.18), $$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \omega_c\sqrt{\mu \in} \quad (2.34)$$

where a, b are in meters.

2. Cutoff Frequency: (f_c)

From equation (2.34), $$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi\sqrt{\mu \in}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$\boxed{f_c = \frac{1}{2\sqrt{\mu \in}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \quad (2.35a)$$

$$\boxed{f_c = \frac{v}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \quad (2.35b)$$

where $v = \frac{1}{\sqrt{\mu \in}}$ = phase velocity of wave in an unbounded dielectric $(\sigma = 0, \mu, \in)$

Cutoff frequency is the operating frequency below which attenuation occurs and above which propagation takes place.

NOTE

f_c depends on
- Dimensions of waveguide (a, b)
- Material inside the waveguide ($\in$, μ)
- Indices of the mode (m, n)
- The rectangular waveguide must be operated at a frequency (f) above cutoff frequency (f_c) for the respective mode to propagate.
- Waveguide acts like high pass filter due to f_c.

3. Cutoff Wavelength: (λ_c)

$$\lambda_c = \frac{v}{f_c}$$

$$\boxed{\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}} \qquad (2.36)$$

NOTE: TM_{11} has longest cutoff wavelength (lowest cutoff frequency) of all TM_{mn} modes.

4. Guide Phase Constant: (β_g)

From equation (2.24b)

$$\beta_g = \sqrt{\omega^2\mu\in - k_c^2}$$

$$= \sqrt{\omega^2\mu\in - \omega_c^2\mu\in}$$

$$\boxed{\beta_g = \omega\sqrt{\mu\in}\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \qquad (2.37a)$$

$$\boxed{\beta_g = \beta\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \qquad (2.37b)$$

where $\beta = \omega\sqrt{\mu\in}$ = Phase constant of wave in an unbounded dielectric $(\sigma = 0, \mu, \in)$

NOTE: β_g must be positive number for propagation i.e. $f > f_c$.

5. Guide Attenuation Constant: (α_g)

Guide propagation constant γ_g when $\boxed{f < f_c}$ is

$$\boxed{\gamma_g = \alpha_g = \beta\sqrt{\left(\frac{f_c}{f}\right)^2 - 1}} \qquad (2.38)$$

where $\beta = \omega\sqrt{\mu\in}$

NOTE: As frequency of operation increases attenuation decreases. $(f\uparrow \Rightarrow \alpha_g\downarrow)$

6. Phase Velocity: (v_p)

The phase velocity of the wave in the guide is given by

$$v_p = \frac{\omega}{\beta_g} = \frac{\omega}{\beta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$\boxed{v_p = \frac{v}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}} \qquad (2.39)$$

where $$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu \in}} = \text{Phase velocity in an unbounded dielectric}$$

NOTE: When $f > f_c$ then $v_p > v$

7. Guide Wavelength: (l_g)

The wavelength λ_g in the guide is given by

$$\boxed{\lambda_g = \frac{2\pi}{\beta_g} = \frac{2\pi}{\beta\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\lambda}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}} \quad (2.40)$$

NOTE: For $f > f_c$, $\lambda_g > \lambda$ where λ is the wavelength in an unbounded dielectric.

8. Wave Impedance: (η_{TM})

The characteristic wave impedance of TM_{mn} modes in the guide can be derived from equation (2.32)

$$\eta_{TM} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\beta_g}{\omega \in} = \beta\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\boxed{\eta_{TM} = \eta\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \eta\frac{\lambda}{\lambda_g}} \quad (2.41)$$

where $$\eta = \frac{\beta}{\omega \in} = \sqrt{\frac{\mu}{\in}} = \text{intrinsic impedance of unbounded dielectric}$$

NOTE

- At $f = f_c$, $\eta_{TM} = 0\ \Omega$
- As f increases (above f_c), η_{TM} value approaches η.
- $\eta_{TM} = 120\pi\frac{\lambda}{\lambda_g}$ (for air dielectric inside guide)

Example - 2.6 In an air-filled waveguide of dimensions a cm × b cm, at a given frequency, the longitudinal component of electric field of TM_{32} mode is of the form $E_z = 20\sin(60\pi x)\sin(100\pi y)$. Which form would E_z have for the lowest order TM mode?

(a) $E_z = 20\sin(20\pi x)$
(b) $E_z = 20\sin(20\pi y)$
(c) $E_z = 20\sin(20\pi x)\sin(50\pi y)$
(d) $E_z = 20\sin(20\pi x)\sin(100\pi y)$

Solution: (c)

The lowest order TM mode will be TM_{11}.

So, $$E_z = 20\sin\left(\frac{60\pi x}{3}\right)\sin\left(\frac{100\pi y}{2}\right)$$

$$= 20\sin(20\pi x)\sin(50\pi y)$$

Example - 2.7 A waveguide section in microwave circuits will act as a

(a) low-pass filter (b) band-pass filter
(c) band-stop filter (d) high-pass filter

Solution:(d)

A waveguide acts as high-pass filter because it allows the transmission of electromagnetic wave above a certain frequency called cut-off frequency.

Example - 2.8 An air filled rectangular waveguide has dimensions a = 6 cm and b = 4 cm. The signal frequency is 6 GHz. Compute the following for TM_{11} mode

(i) Cut-off wave number
(ii) Cut-off frequency
(iii) Cut-off wavelength
(iv) Guide phase constant
(v) Guide attenuation constant
(vi) Guide propagation constant
(vii) Guide wavelength
(viii) Phase velocity
(ix) Wave impedance

Solution:

$$a = 6\text{ cm},\quad b = 4\text{ cm},\quad f = 6\text{ GHz},\quad TM_{11}\ (m=1,\ n=1)$$

(i) Cut-off wave number, $k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \sqrt{\left(\frac{\pi}{6}\right)^2 + \left(\frac{\pi}{4}\right)^2}$

$$k_c = 0.94\text{ rad/cm} = 94\text{ rad/m}$$

(ii) Cut-off frequency, $f_c = \frac{1}{2\pi\sqrt{\mu\in}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \frac{3\times10^{10}}{2\pi}\sqrt{\left(\frac{\pi}{6}\right)^2 + \left(\frac{\pi}{4}\right)^2}$

$$f_c = 4.506\text{ GHz}$$

(iii) Cut-off wavelength (λ_c)

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} = \frac{2}{\sqrt{\frac{1}{36}+\frac{1}{16}}} = 6.66\text{ cm}$$

(iv) Guide phase constant

$$\beta_g = \omega\sqrt{\mu\in}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

Since $f > f_c$ i.e 6 GHz > 4.506 GHz wave propagates and β_g is positive constant

$$\beta_g = \frac{2\pi\times6\times10^9}{3\times10^8}\sqrt{1-\left(\frac{4.506}{6}\right)^2}$$

$$\beta_g = 54.79\text{ rad/m}$$

(v) Guide attenuation constant

Since $f_c < f$, wave propagates with no attenuation.

$$\boxed{\alpha_g = 0}$$

(vi) Guide propagation constant

$$\gamma_g = \alpha_g + j\beta_g = j54.79\ (\text{m}^{-1})$$

(vii) Guide wavelength $\lambda_g = \dfrac{\lambda}{\sqrt{1-\left(\dfrac{\lambda}{\lambda_c}\right)^2}} = \dfrac{\dfrac{3\times10^{10}}{6\times10^{9}}}{0.436} = 11.47\text{ cm}$

NOTE

$$\sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2} = \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

(viii) Phase Velocity $v_p = \dfrac{v}{\sqrt{1-\left(\dfrac{f_c}{f}\right)^2}} = \dfrac{3\times10^8}{0.436} = 6.88\times10^8\text{ m/s}$

(ix) Wave impedance $Z_{TM} = \eta\sqrt{1-\left(\dfrac{f_c}{f}\right)^2} = 120\pi\times0.436$

$Z_{TM} = 164.4\ \Omega$

Example - 2.9 A TM mode operating at 6 GHz is propagated in the air filled waveguide. If

$$E_z = 2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{b}\right)\cos(\omega t - 50z)\text{ v/m}$$

Find

(a) The mode propagating (b) The cut-off frequency

Solution:

(a) E_z in TM mode in general is given by

$$E_z(x, y, z, t) = E_0\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos(\omega t - \beta_g z)$$

NOTE: Any field expression in terms of (x, y, z, t) can be obtained from (x, y, z) expression by multiplying (x, y, z) expression by $e^{j\omega t}$ and taking its real part.

Given $E_z = 2\sin\left(\dfrac{\pi x}{a}\right)\sin\left(\dfrac{2\pi y}{b}\right)\cos(\omega t - 50z)\text{ v/m}$

On comparing, $m = 1,\quad n = 2,\quad \beta_g = 50\text{ rad/m}$

Therefore, $\boxed{TM_{12}\text{ mode}}$ is propagating.

(b) $\beta = \dfrac{\omega}{v} = \dfrac{2\pi\times10^9\times6}{3\times10^8} = 40\pi$

$$\frac{\beta_g}{\beta} = \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow \quad \frac{50}{40\pi} = \sqrt{1-\left(\frac{f_c}{6}\right)^2}$$

$$\Rightarrow \quad f_c = 5.5\text{ GHz}$$