Electronics Engineering

Electromagnetics

Comprehensive Theory

with Solved Examples and Practice Questions

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Electromagnetics

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CHAPTER 1

Vector Analysis

1.1 Introduction

This introductory chapter provides an elegant mathematical language in which electromagnetic (EM) theory is conveniently expressed and best understood. The quantities of interest appearing in the study of EM theory can almost be classified as either a scalar or a vector.

Quantities that can be described by a magnitude alone are called **scalars**. Distance, temperature, mass etc. are examples of scalar quantities. Other quantities, called **vectors**, require both a magnitude and a direction to fully characterize them. Examples of vector quantities include velocity, force, acceleration etc.

In electromagnetics, we frequently use the concept of a **field**. A field is a function that assigns a particular physical quantity to every point in a region. In general, a field varies with both position and time. There are scalar fields and vector fields. Temperature distribution in a room and electric potential are examples of scalar fields. Electric field and magnetic flux density are examples of vector fields.

NOTE: Vectors are denoted by an arrow over a letter ($\vec{A}$) and scalars are denoted by simple letter (A).

1.1.1 Unit Vector

A unit vector $\hat{a}_A$ along $\vec{A}$ is defined as a vector whose magnitude is unity (*i.e.*, 1) and its direction is along $\vec{A}$, that is

$$\hat{a}_A = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A} \quad ...(1.1)$$

Thus we can write $\vec{A}$ as

$$\vec{A} = A\,\hat{a}_A = |\vec{A}|\hat{a}_A \quad ...(1.2)$$

Remember: Any vector can be written as product of its magnitude and its unit vector.

1.1.2 Vector Addition and Subtraction

Two vectors $\vec{A}$ and $\vec{B}$ can be added together to give another vector $\vec{C}$; that is,

$$\vec{C} = \vec{A}+\vec{B} \quad ...(1.3)$$

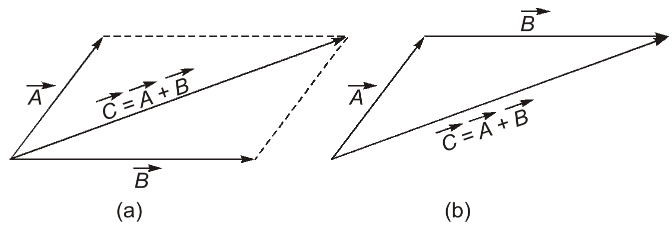


***Figure.* 1.1:** *Vector addition (a) parallelogram rule, (b) head-to-tail rule.*

NOTE

- $\vec{A}+\vec{B}=\vec{B}+\vec{A}$ (Commutative law)
- $(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$ (Associative law)

Vector subtraction is similarly carried out as

$$\vec{D} = \vec{A}-\vec{B}=\vec{A}+(-\vec{B}) \quad ...(1.4)$$

Remember: Graphically, vector addition and subtraction are obtained by either the parallelogram rule or the head-to-tail rule as potrayed in figure 1.1.

NOTE

- $k(\vec{A}+\vec{B})=k\vec{A}+k\vec{B}$ (Distributive law)
- $\dfrac{\vec{A}+\vec{B}}{k}=\dfrac{1}{k}\vec{A}+\dfrac{1}{k}\vec{B}$

1.1.3 Position and Distance Vectors:

A point P in cartesian coordinates may be represented by (x, y, z).

The position vector $\vec{r}_p$ (or radius vector) of point P is defined as the directed distance from origin O to P.

$$\vec{r}_p = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z \quad ...(1.5)$$

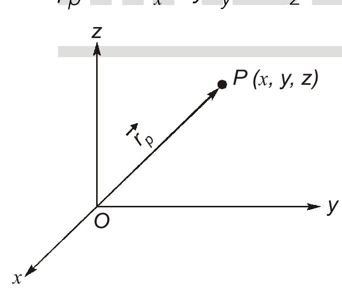


Figure 1.2: *Illustration of position vector* $\vec{r}_p = x\hat{a}_x + \hat{y}a_y + z\hat{a}_z$

The distance vector is the displacement from one point to another.

Consider point P with position vector $\vec{r}_p$ and point Q with position vector $\vec{r}_q$. The displacement from P to Q is written as

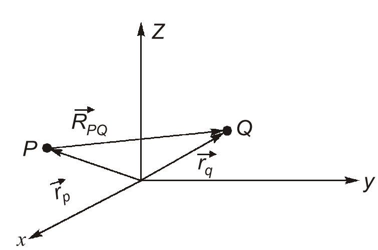


Figure 1.3: *Vector distance* $\vec{R}_{PQ}$

$$\vec{R}_{PQ} = \vec{r}_q - \vec{r}_p \quad ...(1.6)$$

Example 1.1 Point P and Q are located at (0, 2, 4) and (–3, 1, 5). Calculate

(a) The position vector P

(b) The distance vector from P to Q

(c) The distance between P and Q

(d) A vector parallel to PQ with magnitude of 10.

Solution:

(a) $$\vec{r}_p = 0\hat{a}_x + 2\hat{a}_y + 4\hat{a}_z = 2\hat{a}_x + 4\hat{a}_z$$

(b) $$\vec{R}_{PQ} = \vec{r}_q - \vec{r}_p = (-3, 1, 5) - (0, 2, 4) = (-3, -1, 1)$$

$$= -3\hat{a}_x - \hat{a}_y + \hat{a}_z$$

(c) The distance between P and Q is the magnitude of $\vec{R}_{PQ}$; that is

$$d = |\vec{R}_{PQ}| = \sqrt{9+1+1} = 3.317$$

(d) Let the required vector be $\vec{A}$, then

$$\vec{A} = A\hat{a}_A$$

where $A = 10$ is magnitude of $\vec{A}$

and $$\hat{a}_A = \frac{\vec{R}_{PQ}}{|\vec{R}_{PQ}|} = \pm\frac{(-3,-1,1)}{3.317}$$

then $$\vec{A} = \pm\frac{10(-3,-1,1)}{3.317} = \pm(-9.045\,\hat{a}_x - 3.015\,\hat{a}_y + 3.015\,\hat{a}_z)$$

1.1.4 Vector Multiplication

When two vectors are multiplied, the result is either a scalar or a vector depending on how they are multiplied. Thus there are two types of vector multiplication.

1. Scalar (or dot) product : $\vec{A} \cdot \vec{B}$
2. Vector (or cross) product : $\vec{A} \times \vec{B}$

Multiplication or three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ can result in either

3. Scaler triple product : $\vec{A} \cdot (\vec{B} \times \vec{C})$
4. Vector triple product : $\vec{A} \times (\vec{B} \times \vec{C})$

Dot Product:

The dot product, or the scalar product of two vectors $\vec{A}$ and $\vec{B}$, written as $\vec{A} \cdot \vec{B}$ is defined geometrically as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of the angle between them.

Thus $$\vec{A} \cdot \vec{B} = A\,B\cos\theta_{AB} \qquad \text{...(1.7)}$$

Where θ_{AB} is the smaller angle between $\vec{A}$ and $\vec{B}$. The result of $\vec{A} \cdot \vec{B}$ is called either the scalar product because it is scalar, or the dot product due to the dot sign.

If $$\vec{A} = (A_x, A_y, A_z)$$

and $$\vec{B} = (B_x, B_y, B_z)$$

then $$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \qquad \text{...(1.8)}$$

NOTE: Two vectors $\vec{A}$ and $\vec{B}$ are said to be orthogonal (or perpendicular) with each other if $\vec{A}\,.\,\vec{B} = 0$

The dot product obeys the following:

Law	Expression	
Commulative	$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$	...(1.9)
Distributive	$\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B}+\vec{A}.\vec{C}$	...(1.10)
	$\vec{A}\,.\,\vec{A} = \lvert\vec{A}\rvert^2 = \lvert A\rvert^2$	...(1.11)

Also note that:

$$\hat{a}_x\cdot\hat{a}_y = \hat{a}_y\cdot\hat{a}_z = \hat{a}_z\cdot\hat{a}_x = 0 \qquad ...(1.12)$$

$$\hat{a}_x\cdot\hat{a}_x = \hat{a}_y\cdot\hat{a}_y = \hat{a}_z\cdot\hat{a}_z = 1 \qquad ...(1.13)$$

Cross Product:

The cross product of two vectors $\vec{A}$ and $\vec{B}$, written as $\vec{A}\times\vec{B}$, is a vector quantity whose magnitude is the area of the parallelopiped formed by $\vec{A}$ and $\vec{B}$ and is in the direction of advance of the right-handed screw as $\vec{A}$ is turned into $\vec{B}$.

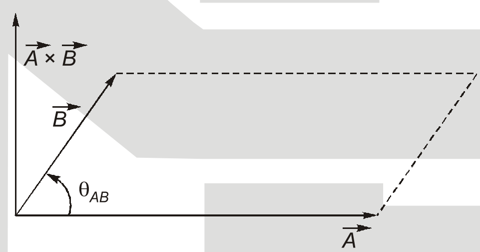


***Figure 1.4:** The cross product of $\vec{A}$ and $\vec{B}$ is a vector with magnitude equal to the area of parallelogram and the direction as indicated*

Thus
$$\vec{A}\times\vec{B} = AB\sin\theta_{AB}\,\hat{a}_n \qquad ...(1.14)$$

where $\hat{a}_n$ is a unit vector normal to the plane containing $\vec{A}$ and $\vec{B}$.

The vector multiplication of equation (1.14) is called **cross product** due to the cross sign. It is also called **vector product** because the result is a vector.

If $\vec{A} = (A_x, A_y, A_z)$ and $B = (B_x, B_y, B_z)$ then :

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad ...(1.15)$$

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{a}_x + (A_zB_x - A_xB_z)\hat{a}_y + (A_xB_y - A_yB_x)\hat{a}_z \qquad ...(1.16)$$

Which is obtained by 'crossing' terms in cyclic permutation, hence the name cross product.

Note that the cross product has the following properties

1. It is not commutative:

$$\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A} \qquad ...(1.17)$$

NOTE: $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$

2. It is not associative:

$$\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C} \qquad ...(1.18)$$

3. It is distributive:

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C} \qquad ...(1.19)$$

NOTE: $\vec{A}\times\vec{A} = 0$

4. Also note that

$$\hat{a}_x\times\hat{a}_y = \hat{a}_z \qquad ...(1.20)$$

$$\hat{a}_y\times\hat{a}_z = \hat{a}_x \qquad ...(1.21)$$

$$\hat{a}_z\times\hat{a}_x = \hat{a}_y \qquad ...(1.22)$$

NOTE: If $\vec{A}\times\vec{B} = 0$, then $\sin\theta_{AB} = 0°$ or $180°$; this shows that $\vec{A}$ and $\vec{B}$ are parallel or antiparallel to each other

Scalar Triple Product:

Given three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, we define scalar triple product as,

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}) \qquad ...(1.23)$$

If $\vec{A} = (A_x, A_y, A_z)$, $\vec{B} = (B_x, B_y, B_z)$ and $\vec{C} = (C_x, C_y, C_z)$, then $\vec{A}\cdot(\vec{B}\times\vec{C})$ is the volume of a parallelopiped having $\vec{A}$, $\vec{B}$, and $\vec{C}$ as edges and is easily obtained by finding the determinant of the 3×3 matrix formed by $\vec{A}$, $\vec{B}$, and $\vec{C}$; that is

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \qquad ...(1.24)$$

Since the result of this vector multiplication is scalar these two equations are called the scalar triple product.

Vector Triple Product:

For vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, we define the vector triple product as

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B}) \qquad ...(1.25)$$

This is obtained using the "bac-cab" rule.

Example 1.2 Given vectors $\vec{A} = 3\hat{a}_x + 4\hat{a}_y + \hat{a}_z$ and $\vec{B} = 2\hat{a}_y - 5\hat{a}_z$, find the angle between $\vec{A}$ and $\vec{B}$

Solution:

The angle θ_{AB} can be found by using either dot product or cross product

$$\vec{A}\cdot\vec{B} = (3, 4, 1) \,.\, (0, 2, -5) = 0 + 8 - 5 = 3$$

$$|\vec{A}| = \sqrt{3^2+4^2+1^2} = \sqrt{26}$$

$$|\vec{B}| = \sqrt{0^2+2^2+5^2} = \sqrt{29}$$

$$\cos\theta_{AB} = \frac{\vec{A}\,.\,\vec{B}}{|\vec{A}|\,|\vec{B}|} = \frac{3}{\sqrt{(26)\times(29)}} = 0.1092$$

$$\theta_{AB} = \cos^{-1}(0.01092) = 83.73°$$

Alternatively:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 3 & 4 & 1 \\ 0 & 2 & -5 \end{vmatrix} = (-20-2)\,\hat{a}_x + (0+15)\,\hat{a}_y + (6-0)\,\hat{a}_z$$

$$= (-22, 15, 6)$$

$$|\vec{A}\times\vec{B}| = \sqrt{(-22)^2+(15)^2+(6)^2} = \sqrt{745}$$

$$\sin\theta_{AB} = \frac{\vec{A}\times\vec{B}}{|\vec{A}|\,|\vec{B}|} = \frac{\sqrt{745}}{\sqrt{(26)\times(29)}} = 0.994$$

$$\theta_{AB} = \sin^{-1}(0.994) = 83.73°.$$

Example 1.3 Three field quantities are given by $\vec{P} = 2\hat{a}_x - \hat{a}_z$ and $\vec{Q} = 2\hat{a}_x - \hat{a}_y + 2\hat{a}_z$, $\vec{R} = 2\hat{a}_x - 3\hat{a}_y + \hat{a}_z$. Determine:

(a) $(\vec{P}+\vec{Q})\times(\vec{P}-\vec{Q})$ (b) $\vec{Q}\,.\,(\vec{R}\times\vec{P})$

(c) $\vec{P}.\,(\vec{Q}\times\vec{R})$ (d) $\sin\theta_{QR}$

(e) $\vec{P}\times(\vec{Q}\times\vec{R})$ (f) A unit vector perpendicular to both $\vec{Q}$ and $\vec{R}$

(g) The component of $\vec{P}$ along $\vec{Q}$

Solution:

(a)

$$(\vec{P}+\vec{Q})\times(\vec{P}-\vec{Q}) = 2(\vec{Q}\times\vec{P})$$

$$= 2\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= 2(1-0)\,\hat{a}_x + 2\,(4+2)\,\hat{a}_y + 2(0+2)\,\hat{a}_z$$

$$= 2\hat{a}_x + 12\hat{a}_y + 4\hat{a}_z$$

(b)

$$\vec{Q}.(\vec{R}\times\vec{P}) = (2, -1, 2)\,.\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= (2, -1, 2).\,(3, 4, 6)$$

$$= 6 - 4, 12 = 14$$

Alternatively:

$$\vec{Q}.(\vec{R}\times\vec{P}) = \begin{vmatrix} 2 & -1 & 2 \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix} = 14$$

(c) $$\vec{P}.(\vec{Q}\times\vec{R}) = \vec{Q}.(\vec{R}\times\vec{P}) = 14$$

(d) $$\sin\theta_{QR} = \frac{|\vec{Q}\times\vec{R}|}{|\vec{Q}|\,|\vec{R}|} = \frac{\sqrt{45}}{3\sqrt{14}} = 0.5976$$

(e) $$\vec{P}\times(\vec{Q}\times\vec{R}) = \vec{Q}(\vec{P}.\vec{R}) - \vec{R}(\vec{P}.\vec{Q})$$
$$= (2, -1, 2)\,(4 + 0 - 1) - (2, -3, -1)\,(4 + 0 - 2)$$
$$= (2, 3, 4)$$
$$= 2\hat{a}_x + 3\hat{a}_y + 4\hat{a}_z$$

(f) A unit vector perpendicular to both $\vec{Q}$ and $\vec{R}$ is given by

$$\hat{a}_n = \pm\frac{\vec{Q}\times\vec{R}}{|\vec{Q}\times\vec{R}|} = \frac{\pm(5,2,-4)}{\sqrt{45}}$$
$$= \pm\,(0.745,\,0.298,\,-0.596)$$
$$\hat{a}_n = \pm\,(0.745\,\hat{a}_x + 0.298\,\hat{a}_y - 0.596\,\hat{a}_z)$$

Note that, $|\hat{a}_n| = 1,\ \hat{a}_n.\vec{Q} = \hat{a}_n.\vec{R} = 0$

The component of $\vec{P}$ along $\vec{Q}$ is

$$\vec{P}_Q = |\vec{P}|\,\cos\theta_{PQ}\,\hat{a}_Q$$
$$= (\vec{P}.\hat{a}_Q)\hat{a}_Q$$
$$= \frac{(\vec{P}.\vec{Q})\vec{Q}}{|\vec{Q}|^2} = \frac{(4+0-2)(2,-1,2)}{(4+1+4)}$$
$$= \frac{2}{9}(2,-1,2)$$
$$= 0.4444\,\hat{a}_x - 0.2222\,\hat{a}_y + 0.4444\,\hat{a}_z$$

1.2 Coordinate Systems

A coordinate system defines points of reference from which specific vector directions may be defined.

Depending on the geometry of the application, one coordinate system may lead to more efficient vector definitions than others. The three most commonly used co-ordinate systems used in the study of electromagnetics are rectangular coordinates (or cartesian coordinates), cylindrical coordinates and spherical coordinates.

NOTE: An orthogonal system is one in which the coordinates are mutually perpendicular

1.2.1 Cartesian Coordinates

A vector $\vec{A}$ in Cartesian (other wise known as rectangular) coordinates can be written as

$$(A_x, A_y, A_z) \text{ or } A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z \qquad ...(1.26)$$

Where a_x, a_y, a_z are unit vectors along the x, y and z directions

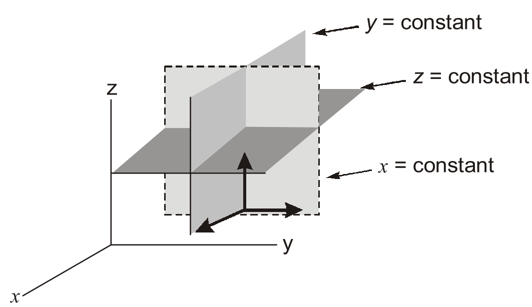


Figure 1.5: *A point in Cartesian coordinates is defined by the intersection of the three planes: x = constant, y = constant, z = constant. The three unit vectors are normal to each of the three surfaces.*

The ranges of the variables are:

$$-\infty \leq x \leq +\infty \quad ...(1.27\text{ a})$$

$$-\infty \leq y \leq +\infty \quad ...(1.27\text{ b})$$

$$-\infty \leq z \leq +\infty \quad ...(1.27\text{ c})$$

1.2.2 Cylindrical Coordinates

The cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry.

A point P in cylindrical coordinates is represented as (ρ, ϕ, z) and is as shown in Fig 1.6. Observe Fig. 1.6 closely and note how we define each space variable; ρ is the radius of the cylinder passing through P or the radial distance from the z-axis; ϕ, called the azimuthal angle, is measured from the x-axis in the xy-plane; and z is the same as in the Cartesian system. The ranges of the variables are:

$$0 \leq \rho \leq \infty \quad ...(1.28)$$

$$0 \leq \phi \leq 2\pi$$

$$-\infty \leq z \leq +\infty$$

A vector $\vec{A}$ in cylindrical coordinates can be written as

$$(A_\rho, A_\phi, A_z) \text{ or } A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z \quad ...(1.29)$$

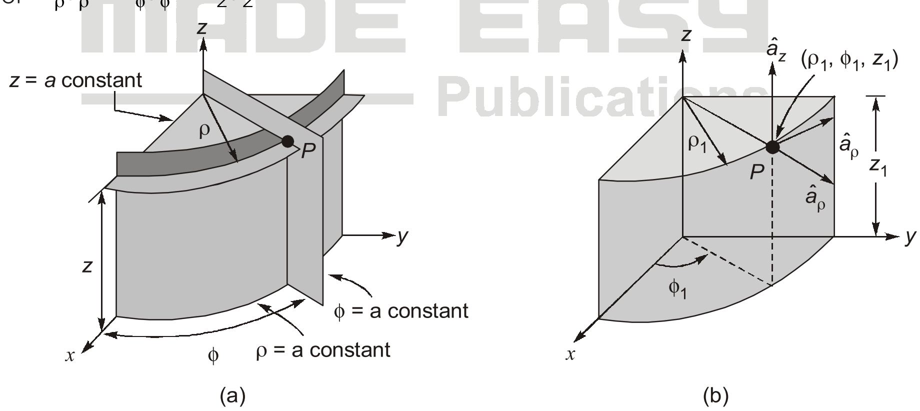


Figure 1.6: *(a) The point is defined by the intersection of the cylinder and the two planes. (b) Point P and unit vectors in the cylindrical coordinate system.*

Notice that the unit vectors $\hat{a}_\rho, \hat{a}_\phi$ and $\hat{a}_z$ are mutually perpendicular because our coordinate system is orthogonal.

$$\hat{a}_\rho \cdot \hat{a}_\phi = \hat{a}_\phi \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_\rho = 0 \quad ...(1.30)$$

$$\hat{a}_\rho \cdot \hat{a}_\rho = \hat{a}_\phi \cdot a_\phi = \hat{a}_z \cdot \hat{a}_z = 1 \quad ...(1.31)$$

$$\hat{a}_\rho \times \hat{a}_\phi = \hat{a}_z \quad ...(1.32)$$

$$\hat{a}_\phi \times \hat{a}_z = \hat{a}_\rho \quad ...(1.33)$$

$$\hat{a}_z \times \hat{a}_\rho = \hat{a}_\phi \quad ...(1.34)$$

The relationships between the variables (x, y, z) of the cartesian coordinate system and those of the cylindrical system (ρ, ϕ, z) are easily obtained from figure 1.7.

Point transformation, $$\rho = \sqrt{x^2+y^2},\ \phi = \tan^{-1}\frac{y}{x},\ z = z \quad ...(1.35)$$

or,

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z \quad ...(1.36)$$

Whereas equation (1.35) is for transforming a point from cartesian (x, y, z) to cylindrical (ρ, ϕ, z) coordinates, equation (1.36) is for (ρ, ϕ, z) $\to$ (x, y, z) transformation.

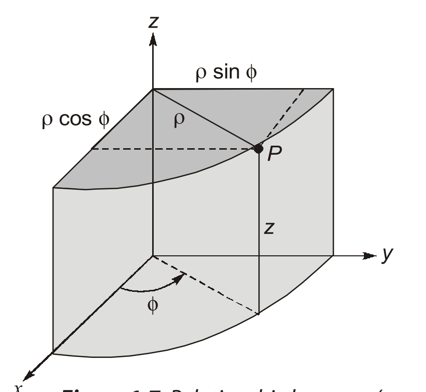


Figure 1.7: *Relationship between (x, y, z) and (ρ, φ, z)*

The relationships between $\hat{a}_x, \hat{a}_y, \hat{a}_z$ and $\hat{a}_\rho, \hat{a}_\phi, \hat{a}_z$ are

Vector transformation, $$\hat{a}_x = \cos\phi\hat{a}_\rho - \sin\phi\hat{a}_\phi \quad ...(1.37\text{ a})$$

$$\hat{a}_y = \sin\phi\hat{a}_\rho + \cos\phi\hat{a}_\phi \quad ...(1.37\text{ b})$$

$$\hat{a}_z = \hat{a}_z \quad ...(1.37\text{ c})$$

or,

$$\hat{a}_\rho = \cos\phi\hat{a}_x + \sin\phi\hat{a}_y \quad ...(1.38\text{ a})$$

$$\hat{a}_\phi = -\sin\phi\hat{a}_x + \cos\phi\hat{a}_y \quad ...(1.38\text{ b})$$

$$\hat{a}_z = \hat{a}_z \quad ...(1.38\text{ c})$$

Finally, the relationship between (A_x, A_y, A_z) and (A_ρ, A_ϕ, A_z) are

$$\begin{vmatrix} A_\rho \\ A_\phi \\ A_z \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} \quad ...(1.39)$$

$$\begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} = \begin{vmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_\rho \\ A_\phi \\ A_z \end{vmatrix} \quad ...(1.40)$$

1.2.3 Spherical Coordinates

The spherical coordinate system is most appropriate when dealing with problems having a degree of spherical symmetry. A point P can be represented as (r, θ, ϕ) and is illustrated in figure. 1.8. From figure. 1.8, we notice that r is defined as the distance from the origin to point P or the radius of a sphere centered at the origin and passing through P; θ (called the colatitudes) is the angle between the z-axis and the position vector of P; and ϕ is measured from the x-axis (the same azimuthal angle in cylindrical coordinates). According to these definitions, the ranges of the variables are

$$0 \le r \le \infty \quad ...(1.41)$$

$$0 \le \theta \le \pi$$

$$0 \le \phi \le 2\pi$$

A vector $\vec{A}$ in spherical coordinates can be written as

(A_r, A_θ, A_ϕ) or $A_r\hat{a}_r + A_\theta\hat{\phi}_\theta + A_\phi\hat{a}_\phi$ ...(1.42)

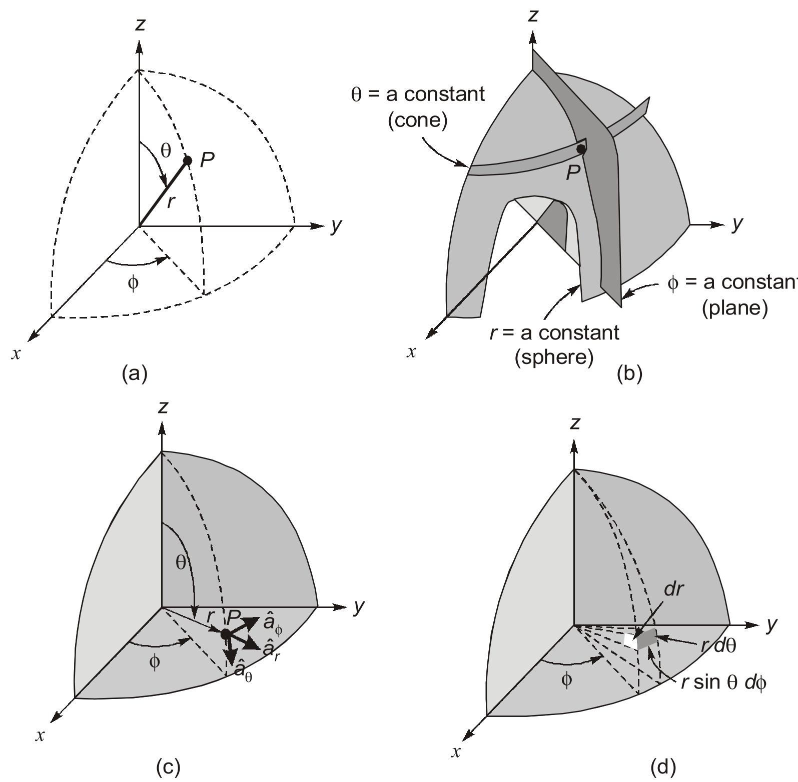


Figure 1.8: *(a) Point P and unit vectors in the cylindrical coordinate system.*
(b) The three mutually perpendicular surfaces of the spherical coordinate system.
(c) The three unit vectors of spherical coordinates.
(d) The differential volume element in the spherical coordinate system.

NOTE: The unit vectors $\hat{a}_r$, $\hat{a}_\theta$, and $\hat{a}_\phi$ are mutually perpendicular because our coordinate system is orthogonal.

$$\hat{a}_r \cdot \hat{a}_\theta = \hat{a}_\theta \cdot \hat{a}_\phi = \hat{a}_\phi \cdot a_r = 0 \quad ...(1.43)$$

$$\hat{a}_r \cdot \hat{a}_r = \hat{a}_\theta \cdot a_\theta = \hat{a}_\phi \cdot \hat{a}_\phi = 1 \quad ...(1.44)$$

$$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi \quad ...(1.45)$$

$$\hat{a}_\theta \times \hat{a}_\phi = \hat{a}_r \quad ...(1.46)$$

$$\hat{a}_\phi \times \hat{a}_r = \hat{a}_\theta \quad ...(1.47)$$

The relationship between the variables (x, y, z) of the cartesian coordinate system and those of the spherical coordinate system (ρ, θ, ϕ) are easily obtained from figure 1.8.

Point transformation, $r = \sqrt{x^2 + y^2 + z^2}$, $\theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z}$, $\phi = \tan^{-1}\dfrac{y}{x}$ ...(1.48)

or $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ ...(1.49)

The relationship between $\hat{a}_x, \hat{a}_y, \hat{a}_z$ and $\hat{a}_r, \hat{a}_\theta, \hat{a}_\phi$ are

$$\hat{a}_x = \sin\theta\cos\phi\hat{a}_r + \cos\theta\cos\phi\hat{a}_\theta - \sin\phi\hat{a}_\phi \quad ...(1.50\text{ a})$$

$$\hat{a}_y = \sin\theta\sin\phi\hat{a}_r + \cos\theta\sin\phi\hat{a}_\theta + \cos\phi\hat{a}_\phi \quad ...(1.50\text{ b})$$

$$\hat{a}_z = \cos\phi\hat{a}_r - \sin\phi\hat{a}_\phi \qquad ...(1.50\ c)$$

or,
$$\hat{a}_r = \sin\theta\cos\phi\hat{a}_x + \sin\theta\sin\phi\hat{a}_y + \cos\phi\hat{a}_z \qquad ...(1.51\ a)$$

$$\hat{a}_\theta = \cos\theta\cos\phi\hat{a}_x + \cos\theta\sin\phi\hat{a}_y - \sin\theta\hat{a}_z \qquad ...(1.51\ b)$$

$$\hat{a}_\phi = -\sin\phi\hat{a}_x + \cos\phi\hat{a}_y \qquad ...(1.51\ c)$$

Finally, the relationship between (A_x, A_y, A_z) and (A_r, A_θ, A_ϕ) are

Vector transformation,
$$\begin{vmatrix} A_r \\ A_\theta \\ A_\phi \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} \qquad ...(1.52)$$

or,
$$\begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\cos\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix} \begin{vmatrix} A_r \\ A_\theta \\ A_\phi \end{vmatrix}$$

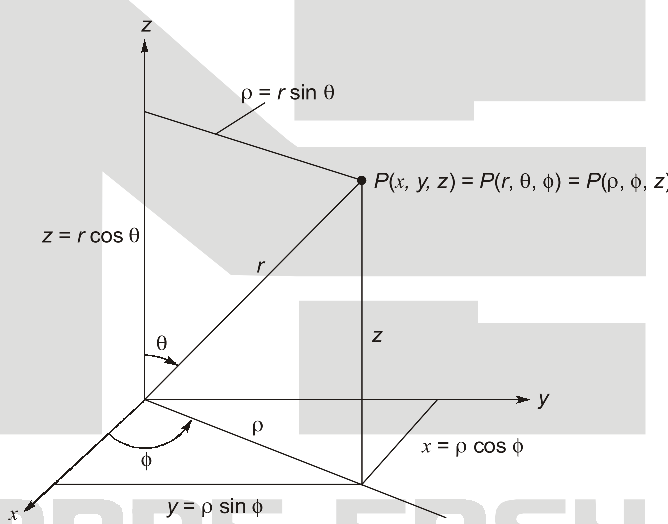


***Figure 1.9:** Relationships between space variables (x, y, z), (r, θ, φ) and (ρ, φ, z)*

Example 1.4 Write an expression for a position vector at any point in space in the rectangular coordinate system. Then transform the position vector into a vector in the cylindrical coordinate system.

Solution:

The position vector of any point $P(x, y, z)$ in space is

$$\vec{A} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$$

Using the tranformation matrix as given in equation (1.39), we obtain

$$A_\rho = x\cos\phi + y\sin\phi$$

$$A_\phi = -x\sin\phi + y\cos\phi \text{ and } A_z = z$$

Substituting $x = \rho\cos\phi$ and $y = \rho\sin\phi$, we obtain

$$A_\rho = \rho,\ A_\phi = 0, \text{ and } A_z = z$$

Thus, the position vector $\vec{A}$ in the cylindrical coordinate system is

$$\vec{A} = \rho\hat{a}_\rho + z\hat{a}_z$$

Example 1.5 Given point $P(-2, 6, 3)$ and vector $\vec{A} = y\hat{a}_x + (x+z)\hat{a}_y$, express P and $\vec{A}$ in cylindrical and spherical coordinates. Evaluate A at P in the cartesian, cylindrical, and spherical systems.

Solution:

At point P: $x = -2,\ y = 6,\ z = 3$. Hence,

$$\rho = \sqrt{x^2+y^2} = \sqrt{4+36} = 6.32$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{6}{-2} = 108.43°$$

$$z = 3$$

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{4+36+9} = 7$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\frac{\sqrt{4+36}}{3} = 64.62°$$

Thus,

$$P(-2, 6, 3) = P(6.32, 108.43°, 3) = P(7, 64.62°, 108.43°)$$

In the cartesian system, $\vec{A}$ at P is

$$\vec{A} = 6\hat{a}_x + \hat{a}_y$$

For $\vec{A}$, $A_x = y$, $A_y = x + z$, $A_z = 0$. Hence, in the cylindrical system

$$\begin{vmatrix} A_\rho \\ A_\phi \\ A_z \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} y \\ x+z \\ 0 \end{vmatrix}$$

or

$$A_\rho = y\cos\phi + (x+z)\sin\phi$$

$$A_\phi = -y\sin\phi + (x+z)\cos\phi$$

$$A_z = 0$$

But $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$, and substituting these yields

$$\vec{A} = (A_\rho, A_\phi, A_z) = [\rho\sin\phi\cos\phi + (\rho\cos\phi + z)\sin\phi]\,\hat{a}_\rho$$

$$+ [-\rho\sin^2\phi + (\rho\cos\phi + z)\cos\phi]\,\hat{a}_\phi$$

At P

$$\rho = \sqrt{40}, \qquad \tan\phi = \frac{6}{-2}$$

Hence,

$$\cos\phi = \frac{-2}{\sqrt{40}}, \qquad \sin\phi = \frac{6}{\sqrt{40}}$$

$$A = \left[\sqrt{40}\cdot\frac{-2}{\sqrt{40}}\cdot\frac{6}{\sqrt{40}} + \left(\sqrt{40}\cdot\frac{-2}{\sqrt{40}} + 3\right)\cdot\frac{6}{\sqrt{40}}\right]\hat{a}_\rho + \left[-\sqrt{40}\cdot\frac{36}{40} + \left(\sqrt{40}\cdot\frac{-2}{\sqrt{40}} + 3\right)\cdot\frac{-2}{\sqrt{40}}\right]\hat{a}_\phi$$

$$= \frac{-6}{\sqrt{40}}\hat{a}_\rho - \frac{38}{\sqrt{40}}\hat{a}_\phi = -0.9487\hat{a}_\rho - 6.008\hat{a}_\phi$$

Similarly, in the spherical system

$$\begin{vmatrix} A_r \\ A_\theta \\ A_\phi \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} \begin{vmatrix} y \\ x+z \\ 0 \end{vmatrix}$$

or,
$$A_r = y\sin\theta\cos\phi + (x+z)\sin\theta\sin\phi$$
$$A_\theta = y\cos\theta\cos\phi + (x+z)\cos\theta\sin\phi$$
$$A_\phi = -y\sin\phi + (x+z)\cos\phi$$

But,
$$x = r\sin\theta\cos\phi,$$
$$y = r\sin\theta\sin\phi,$$
$$z = r\cos\theta,$$

and substituting these yields
$$\vec{A} = (A_r, A_\theta, A_\phi)$$
$$= r[\sin^2\theta\cos\phi\sin\phi + (\sin\theta\cos\phi + \cos\theta)\sin\theta\sin\phi]\ \hat{a}_r$$
$$+ r\{\sin\theta\cos\theta\sin\phi\cos\phi + (\sin\theta\cos\phi + \cos\theta)\cos\theta\sin\phi]\,\hat{a}_\theta$$
$$+ r[-\sin\theta\sin^2\phi + (\sin\theta\cos\phi + \cos\theta)\cos\phi]\,\hat{a}_\phi$$

At P
$$r = 7,\ \tan\phi = \frac{6}{-2},\qquad \tan\theta = \frac{\sqrt{40}}{3}$$

Hence,
$$\cos\phi = \frac{-2}{\sqrt{40}},\qquad \sin\phi = \frac{6}{\sqrt{40}},\qquad \cos\theta = \frac{3}{7},\qquad \sin\theta = \frac{\sqrt{40}}{7}$$

$$\vec{A} = 7\cdot\left[\frac{40}{49}\cdot\frac{-2}{\sqrt{40}}\cdot\frac{6}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{\sqrt{40}}{7}\cdot\frac{6}{\sqrt{40}}\right]\hat{a}_r$$

$$+7\cdot\left[\frac{\sqrt{40}}{49}\cdot\frac{3}{7}\cdot\frac{6}{\sqrt{40}}\cdot\frac{-2}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{3}{7}\cdot\frac{6}{\sqrt{40}}\right]\hat{a}_\theta + 7\cdot\left[\frac{-\sqrt{40}}{49}\cdot\frac{36}{40} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{-2}{\sqrt{40}}\right]\hat{a}_\phi$$

$$= \frac{-6}{7}\hat{a}_r, -\frac{18}{7\sqrt{40}}\hat{a}_\theta - \frac{38}{\sqrt{40}}\hat{a}_\phi$$

$$= -0.8571\,\hat{a}_r\ -0.4066\,\hat{a}_\theta - 6.008\,\hat{a}_\phi$$

Note that $|A|$ is the same in the three systems; that is,
$$|A(x, y, z)| = |A(\rho, \phi, z)| = |A(r, \theta, \phi)| = 6.083.$$

1.3 Vector Calculus

1.3.1 Introduction

The first section is mainly focused on vector addition, subtraction, and multiplication in cartesian coordinates, and the second section extended all these to other coordinate systems. This chapter deals with vector calculus-integration and differentiation of vectors. The concepts introduced in this section provide a convenient language for expressing certain fundamental ideas in electromagnetics in general.

1.3.2 Differential Length, Area and Volume

In our study of electromagnetism we will often be required to perform line, surface, and volume integrations. The evaluation of these integrals in a particular coordinate system requires the knowledge of differential elements of length, surface, and volume. In the following subsections we describe how these differential elements are constructed in each coordinate system.

Cartesian Coordinates

From figure 1.10, we notice that:

1. Differential displacement is given by:

$$\vec{dl} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z \qquad ...(1.54)$$

2. Differential normal area is given by:

$$\vec{dS} = \begin{cases} dydz\hat{a}_x \\ dxdz\hat{a}_y \\ dxdy\hat{a}_z \end{cases} \qquad ...(1.55a)$$

3. Differential volume is given by:

$$dv = dxdydz \qquad ...(1.55b)$$

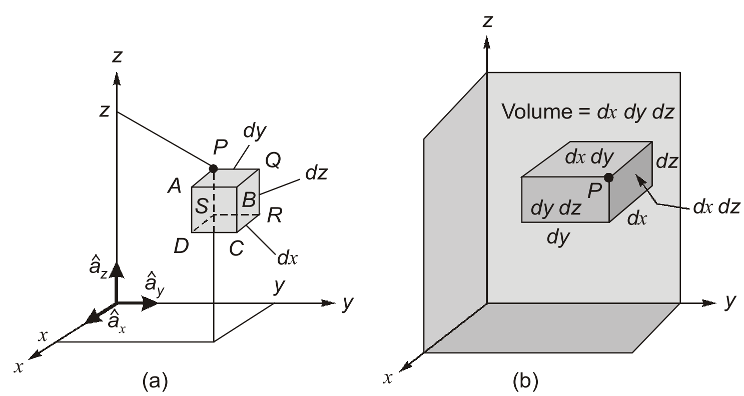


***Figure 1.10 :** Differential elements in the right-handed cartesian coordinate system*

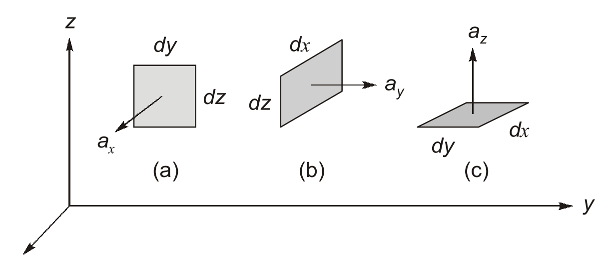


***Figure 1.11 :** Differential normal areas in Cartesian coordinates.*

The way $\vec{dS}$ is defined is important. The differential surface (or area) element $\vec{dS}$ may generally be defined as:

$$\vec{dS} = dS\hat{a}_n \qquad ...(1.57)$$

where dS is the area of the surface element and $\hat{a}_n$ is a unit vector normal to the surface dS (and directed away) from the volume if dS is part of the surface describing a volume). If we consider surface *ABCD* in figure 1.10, for example, $\vec{dS} = dydz\hat{a}_x$ whereas for surface *PQRS*, $\vec{dS} = -dydz\hat{a}_x$ because $\hat{a}_n = -\hat{a}_x$ is normal to *PQRS*.

Remember: What we have to remember at all times about differential elements is $\vec{dl}$ and how to get $\vec{dS}$ and dv from it. Once dl is remembered, $\vec{dS}$ and dv can easily be found.

Cylindrical Coordinates

From figure 1.12, we notice that:

1. Differential displacement is given by:

$$\vec{dl} = d\rho\hat{a}_\rho + \rho d\phi\hat{a}_\phi + dz\hat{a}_z \qquad ...(1.58)$$

2. Differential normal area is given by:

$$\vec{dS} = \begin{cases} \rho d\phi dz\hat{a}_\rho \\ d\rho dz\hat{a}_\phi \\ \rho d\phi d\rho\hat{a}_z \end{cases} \qquad ...(1.59)$$

3. Differential volume is given by:

$$dv = \rho d\rho d\phi dz \qquad ...(1.60)$$

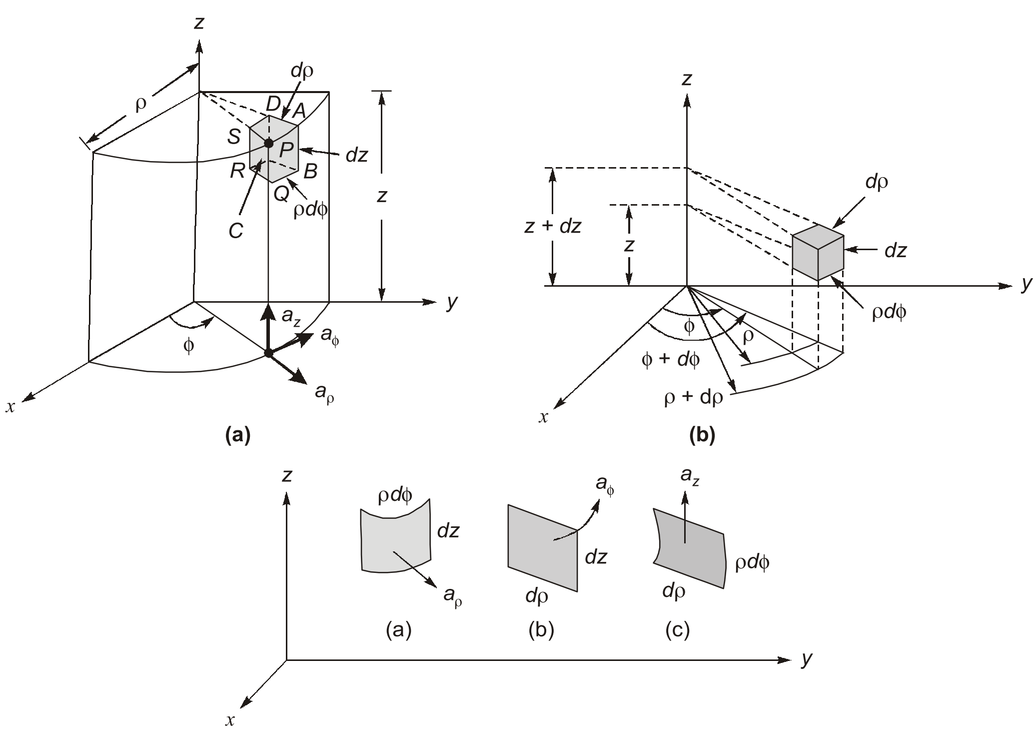


Figure 1.12: *(a) (b) Differential elements in cylindrical coordinates*
(c) Differential normal areas in cylindrical coordinates

Spherical Coordinates

From figure 1.13, we notice that:

1. Differential displacement is given by:

$$\vec{dl} = dr\hat{a}_r + rd\theta\hat{a}_\theta + r\sin\theta d\phi\hat{a}_\phi \qquad ...(1.61)$$

2. Differential normal area is given by:

$$\vec{dS} = \begin{cases} r^2\sin\theta d\theta d\phi\hat{a}_r \\ r\sin\theta dr d\phi\hat{a}_\theta \\ r\,dr\,d\theta\hat{a}_\phi \end{cases} \qquad ...(1.62)$$

3. Differential volume is given by:

$$dv = r^2 \sin\theta\ dr\ d\theta\ d\phi \qquad ...(1.63)$$

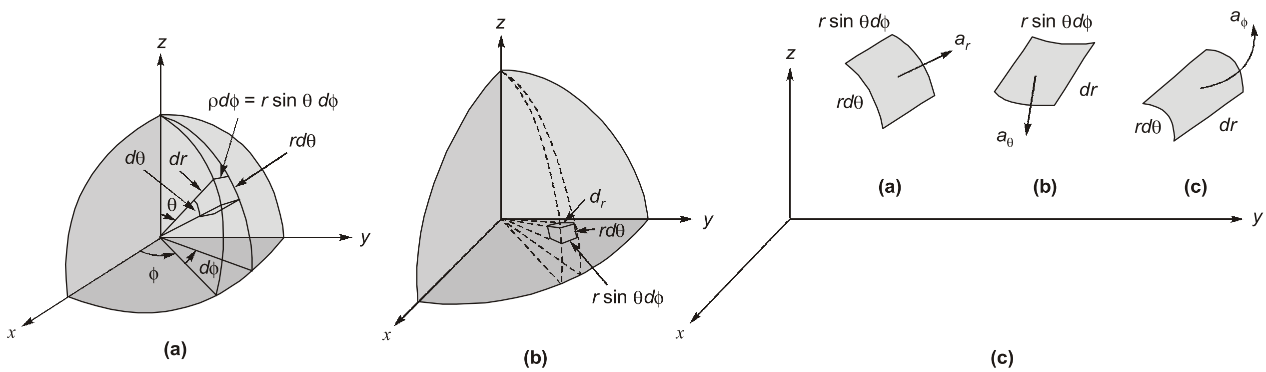


Figure. 1.13: *(a) (b) Differential elements in spherical coordinates.*
(c) Differential normal areas in spherical coordinates

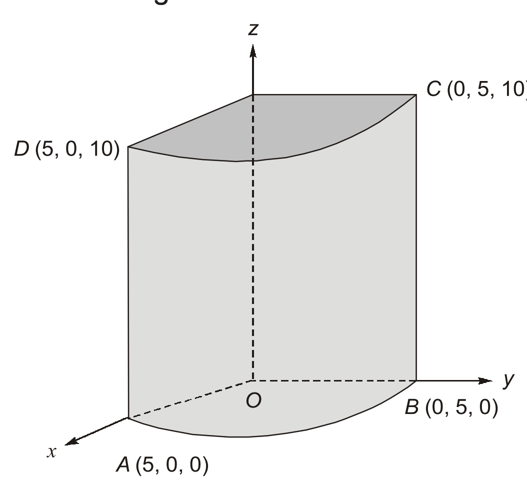
For easy reference, the differential length, surface, and volume elements for the three coordinate systems are summarized in Table 1.1.

Table 1.1 : Differential elemnts of length, surface, and volume in the rectangular, cylindrical, and spherical coordinate systems

Differential elements	Coordinate system		
	Rectangular (Cartesian)	**Cylindrical**	**Spherical**
Length $d\vec{l}$	$dx\,\vec{a}_x$ $+dx\,\vec{a}_y$ $+dz\,\vec{a}_z$	$d\rho\,\vec{a}_\rho$ $+\rho d\phi\,\vec{a}_\phi$ $+dz\,\vec{a}_z$	$dr\,\vec{a}_r$ $+r\,d\phi\,\vec{a}_\theta$ $+r\sin\theta\,d\phi\,\vec{a}_\phi$
Surface $d\vec{s}$	$dy\,dz\,\vec{a}_x$ $+dx\,dz\,\vec{a}_y$ $+dx\,dy\,\vec{a}_z$	$\rho d\phi\,dz\,\vec{a}_\rho$ $+d\rho\,dz\,\vec{a}_\phi$ $\rho d\rho\,d\phi\,\vec{a}_z$	$r^2\sin\theta\,d\theta\,\vec{a}_r$ $+r\,dr\sin\theta\,d\phi\,\vec{a}_\theta$ $+r\,dr\;d\theta\,\vec{a}_\phi$
Volume dv	$dx\,dy\,dz$	$\rho d\rho\,d\phi\,dz$	$r^2 dr\sin\theta\,d\theta\,d\phi$

Example 1.6 Consider the object shown in figure below. Calculate:

1. The distance *BC*.
2. The distance *CD*.
3. The surface area *ABCD*.
4. The surface area *ABO*.
5. The surface area *AOFD*.
6. The volume *ABDCFO*.



Solution :

Although points *A*, *B*, *C* and *D* are given in cartesian coordinates, it is obvious that the object has cylindrical symmentry. Hence, we solve the problem in cylindrical coordinates. The points are transformed from cartesian to cylindrical coordinates as follows:

$$A(5, 0, 0) \to A\,(5, 0°, 0)$$

$$B\,(0, 5, 0) \to A\left(5, \frac{\pi}{2}, 0\right)$$

$$C\,(0, 5, 10) \to C\left(5, \frac{\pi}{2}, 10\right)$$

$$D\,(5, 0, 10) \to D\,(5, 0°, 10)$$

1. Along BC, $dl = dz$; hence,

$$BC = \int dl = \int_0^{10} dz = 10$$

2. Along CD, $dl = \rho d\phi$ and $\rho = 5$, so

$$CD = \int_0^{\pi/2} \rho\, d\phi = 5\phi\Big|_0^{\pi/2} = 2.5\pi$$

3. For $ABCD$, $dS = \rho d\phi dz$, $\rho = 5$. Hence

$$\text{area } ABCD = \int dS = \int_{\phi=0}^{\pi/2}\int_{z=0}^{10} \rho\, d\phi\, dz = 5\int_{\phi=0}^{\pi/2} d\phi \int_{z=0}^{10} dz = 25\pi$$

4. For ABO, $dS = \rho d\phi d\rho$ and $z = 0$, so

$$\text{area } ABO = \int_{\phi=0}^{\pi/2}\int_{\rho=0}^{5} \rho\, d\phi\, d\rho = \int_{\phi=0}^{\pi/2} d\phi \int_0^5 \rho\, d\rho = 6.25\pi$$

5. For $AOFD$, $dS = d\rho dz$ and $\phi = 0°$, so

$$\text{area } AOFD = \int_{\rho=0}^{5}\int_{z=0}^{10} d\rho\, dz = 50$$

6. For volume $ABDCFO$, $dv = \rho d\phi\, dz\, d\rho$

$$v = \int dv = \int_{\rho=0}^{5}\int_{\phi=0}^{\pi/2}\int_{z=0}^{10} \rho\, d\phi\, dz\, d\rho = \int_0^{10} dz \int_0^{\pi/2} d\phi \int_0^5 \rho\, d\rho = 62.5\pi$$

1.3.3 Line, Surface, and Volume Integrals

Line Integral

The familiar concept of integration will now be extended to cases when the integrand involves a vector. By a line we mean the path along a curve in space. We shall use terms such as line, curve, and contour interchangeably.

> The line integral $\int_L \vec{A}.\vec{dl}$ is the integral of the tangential component of $\vec{A}$ along curve L.

Given a vector field $\vec{A}$ and a curve L, we define the integral as the line integral of $\vec{A}$ around L (see figure 1.14):

$$\int_L \vec{A}\cdot\vec{dl} = \int_a^b |\vec{A}|\cos\theta\, dl \qquad \text{...(1.64)}$$

If the path of integration is a closed curve such as abca in figure 1.15, precedent equation becomes a closed contour integral.

$$\oint_L \vec{A}\cdot\vec{dl} \qquad \text{...(1.65)}$$

Which is called the circulation of $\vec{A}$ around L.

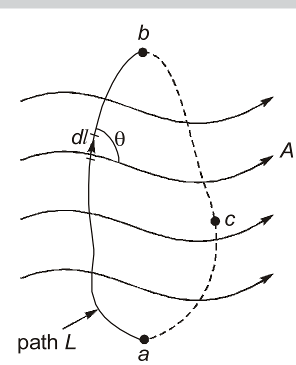


***Figure 1.14 :** Path of integration of vector field A.*