Electrical Engineering

Signals and Systems

Comprehensive Theory

with **Solved Examples** and **Practice Questions**

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Signals and Systems

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Signals and Systems

Introduction to Signals and Systems

This book starts with basic and extensive chapter on signals in which continuous and discrete-time case are discussed in parallel. A variety of basic signals, functions with their mathematical description, representation and properties are incorporated. A substantial amount of examples are given for quick sketching of functions. A chapter on systems is discussed separately which deals with classification of systems, both in continuous and discrete domain and more emphasize is given to LTI systems and analytical as well as graphical approach is used to understand convolution operation. These two chapters makes backbone of the subject.

Further we shall proceed to transform calculus which is important tool of signal processing. A logical and comprehensive approach is used in sequence of chapters. The continuous time Fourier series which is base to the Fourier transform, deals with periodic signal representation in terms of linear complex exponential, is discussed.

The Fourier transform is discussed before Laplace transform. The sampling, a bridge between continuous-time and discrete-time, is discussed to understand discrete-time domain.

A major emphasis is given on proof of the properties so that students can understand and analyzes fundamental easily.

A point wise recapitation of all the important points and results in every chapter proves helpfull to students in summing up essential developments in the chapter which is an integral part of any competitive examination.

CHAPTER 1

Introduction to Signals

Introduction

A signal is any quantity having information associated with it. It may also be defined as a function of one or more independent variables which contain some information.

A function defines a relationship between two sets i.e. one is domain and another is range.

It means function defines mapping from one set to another and similarly a signal may also be defined as mapping from one set (domain) to another (range). e.g.

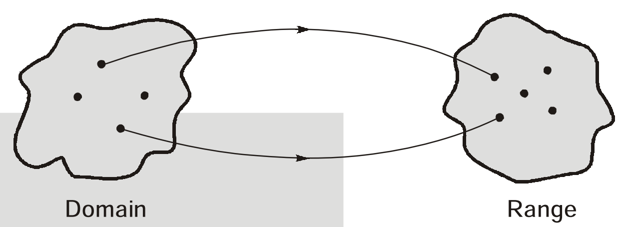


Figure-1.1

- A speech signal would be represented by acoustic pressure as a function of time.
- A monochromatic picture would be represented by brightness as a function of two spatial variable.
- A voltage signal is defined by a voltage across two points varying as function of time.
- A video signal, in which color and intensity as a function of 2-dimensional space (2D) and 1-dimensional time (i.e. hybrid variables).

Note: In this course of "signals and systems", we shall focus on signals having only one variable and will consider 'time' as independent variable.

1.1 Elementary Signals

These signals serve as basic building blocks for construction of somewhat more complex signals. The list of elementary signals mainly contains singularity functions and exponential functions.

These elementary signals are also known as basic signals/standard signals.

Let us discuss these basic signals one-by-one.

1.1.1 Unit Impulse Function:

A continuous-time unit impulse function δ(t), also called as dirac delta function is defined as

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

The unit-impulse function is represented by an arrow with strength of ‘1’ which represents its ‘area’ or ‘weight’.

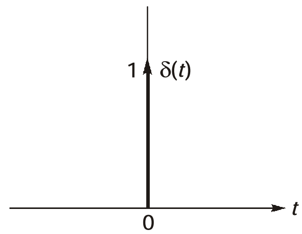


Figure-1.2

The above definition of an impulse function is more generalised and can be represented as limiting process without any regard to shape of a pulse. For example, one may define impulse function as a limiting case of rectangular pulse, triangular pulse Gaussian pulse, exponential pulse and sampling pulse as shown below:

(i) Rectangular Pulse:

$$\delta(t) = \lim_{\varepsilon \to 0} p(t)$$

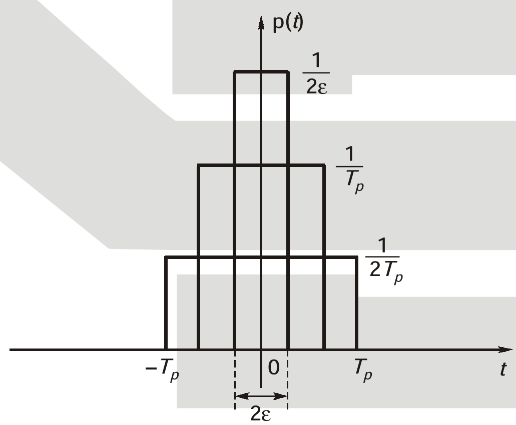


Figure-1.3

(ii) Triangular Pulse:

$$\delta(t) = \begin{cases} \lim_{\tau \to 0} \dfrac{1}{\tau}\left[1 - \dfrac{|t|}{\tau}\right] & ; \quad |t| < \tau \\ 0 & ; \quad |t| > \tau \end{cases}$$

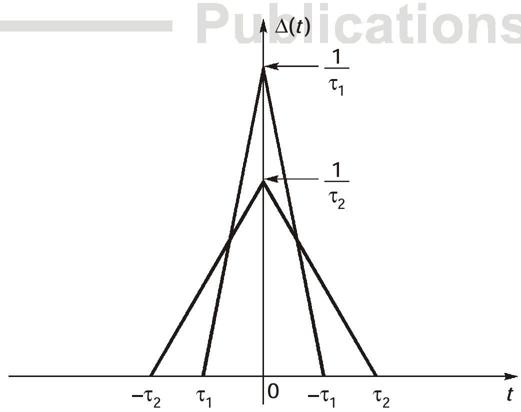


Figure-1.4

CHAPTER 5

Laplace Transform

Introduction

In Fourier transform, one problem we faced that Fourier transform is not applicable for signals that are not absolutely integrable. This problem could be resolved by generalizing Fourier transform which leads to development of Laplace transform. The Laplace transform can be applied to broader class of signals compared to Fourier transform and analysis of many unstable systems thus play an important role in study of stability of systems.

There are two varieties of Laplace transform: bilateral and unilateral. The bilateral one, also known as two- sided Laplace transform is defined for $-\infty \le t \le \infty$ can handle all causal and non causal signals. It provides insights about system's characteristics such as stability, causality and frequency response.

The unilateral one, also known as one-sided Laplace transform can handle only causal signals and mainly used to solve differential equations with initial conditions.

5.1 The Definition

Consider a continuous time signal $x(t)$. Its Laplace transform (bilateral) is defined as

$$LT\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\cdot e^{-st}dt$$

Where 's' is complex frequency which is given as

$$s = \sigma + j\omega$$

where, σ = Re $\{s\}$ is damping factor

ω = Im $\{s\}$ is oscillation frequency in 'rad/s'

The transform $X(s)$ is (complex) frequency domain specification of $x(t)$. Thus $x(t)$ and $X(s)$ makes Laplace transform pair as

$$\underbrace{x(t)}_{\substack{\text{Continuous}\\\text{Aperiodic}}} \xleftrightarrow{\text{LT}} \underbrace{X(s)}_{\substack{\text{Aperiodic}\\\text{continuous}}}$$

and the inverse Laplace transform of $X(s)$ is given as

$$ILT\{X(s)\} = LT^{-1}[X(s)] = x(t)$$

$$x(t) = \frac{1}{2\pi j}\int_{C-j\infty}^{C+j\infty} X(s)e^{st}ds$$

where 'c' is constant and responsible for convergence of integral.

5.2 Relationship between Laplace Transform and Fourier Transform

For a signal $x(t)$, its Laplace transform $X(s)$ is given by

$$L[x(t)] = X(s) = \int_{-\infty}^{+\infty} x(t)\cdot e^{-st}dt$$

put $s = \sigma + j\omega$, we get

$$= \int_{-\infty}^{+\infty} x(t)e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{+\infty} \left[x(t)\cdot e^{-\sigma t}\right]e^{-j\omega t}dt$$

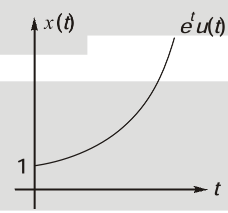
$$= FT\left[x(t)e^{-\sigma t}\right]$$

Therefore, $$L[x(t)] = FT\left[x(t)\cdot e^{-\sigma t}\right]$$

at $\sigma = 0$, $s = j\omega$

$$L[x(t)] = FT[x(t)]$$

Example - 5.1 For the signal shown below:



(a) Only Fourier transform exists
(b) Only Laplace transform exists
(c) Both Laplace and Fourier transforms exist
(d) Neither Laplace nor Fourier transform exists

Solution: (b)

5.3 Eigen Value and Eigen Function

Let $x(t) = e^{st}$, complex exponential is applied to LTI system with impulse response $h(t)$. The output of system is given by

$$y(t) = h(t) * x(t) \quad \text{(i)}$$

$$= \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)\cdot e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

$$y(t) = e^{st}\cdot H(s) \quad \text{...(ii)}$$

Where, $$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\, d\tau = \int_{-\infty}^{+\infty} h(t)e^{-st}\, dt$$

$H(s)$ is the Laplace transform of impulse response $h(t)$ is known as transfer function of the LTI systems.For a signal, if system output is constant times the input then input is referred to as an eigen function of system and amplitude factor is referred as eigen value of the system.

Thus "e^{st}" is an eigen function and $H(s)$ as the corresponding eigen value of the system.

Example - 5.3 Consider the following statements:

1. The Laplace transform of the unit impulse function is s × Laplace transform of the unit ramp function.
2. The impulse function is a time derivative of the ramp function.
3. The Laplace transform of the unit impulse function is s × Laplace transform of the unit step function.
4. The impulse function is a time derivative of the unit step function.

Which of these statements is/are correct?

(a) 1 and 2 only (b) 3 and 4 only
(c) 2 and 3 only (d) 1, 2, 3 and 4

***Solution:* (b)**

$$\delta(t) = \frac{d}{dt}u(t)$$

$$\therefore \quad LT(\delta(t)) = sLT(u(t))$$

Example - 5.4 Find the Laplace transform of following signals and corresponding ROC.

(i) $x(t) = e^{-t}\,u(t) + e^{-2t}\,u(t)$ (ii) $x(t) = e^{-t}\,u(t) + e^{2t}\,u(-t)$ (iii) $x(t) = e^{t}\,u(t) + e^{-2t}\,u(-t)$

Solution:

(i)

$$x(t) = e^{-t}\,u(t) + e^{-2t}\,u(t)$$

$$e^{-t}u(t) \xleftrightarrow{LT} \frac{1}{s+1}, \quad \text{Re}\{s\} > -1$$

$$e^{-2t}u(t) \xleftrightarrow{LT} \frac{1}{s+2}, \quad \text{Re}\{s\} > -2$$

Hence, common ROC exists and it is right of right most pole in s-plane and it is Re$\{s\} > -1$

$$X(s) = \left[\frac{1}{s+1} + \frac{1}{s+2}\right], \quad \text{Re}\{s\} > -1$$

$$X(s) = \frac{2s+3}{(s+1)(s+2)}, \quad \text{Re}\{s\} > -1$$

(ii)

$$x(t) = e^{-t}u(t) + e^{2t}u(-t)$$

$$e^{-t}u(t) \xleftrightarrow{LT} \frac{1}{s+1}, \quad \text{Re}\{s\} > -1$$

$$e^{2t}u(-t) \xleftrightarrow{LT} \frac{-1}{s-2}, \quad \text{Re}\{s\} < 2$$

Hence common ROC exists and it is $-1 < \text{Re}\{s\} < 2$

$$X(s) = \left[\frac{1}{s+1} + \frac{-1}{s-2}\right], \quad -1 < \text{Re}\{s\} < 2$$

$$X(s) = \frac{-3}{(s+1)(s-2)}, \quad -1 < \text{Re}\{s\} < 2$$

Student's Assignments 1

Objective Questions

Q.1 Laplace transform of double differentiation of unit impulse signal is

(a) 1 (b) s

(c) s^2 (d) $\dfrac{1}{s^2}$

Q.2 A continuous time signal

$$x(t) = [t^{-1}e^{-3t} - t^{-1}e^{-2t}]u(t)$$

has zero initial conditions. Laplace transform of signal $x(t)$ is

(a) $\log(s+3) - \log(s+2)$

(b) $\log(s+3) + \log(s+2)$

(c) $-\log(s+3) - \log(s+2)$

(d) $\log(s+2) - \log(s+3)$

Q.3 Signals $x(t)$ and $y(t)$ have the following pole-zero diagrams

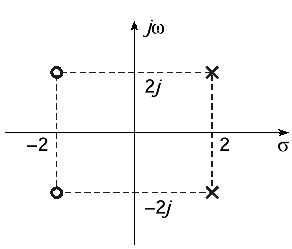


figure : $x(t)$

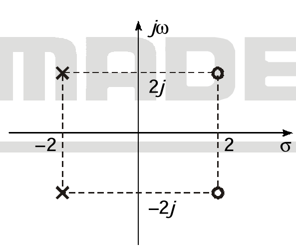


figure : $y(t)$

The signals $g(t)$ and $h(t)$ are defined as $g(t) = x(t)e^{-3t}$ and $h(t) = y(t) * e^{-t}u(t)$.

If $g(t)$ and $h(t)$ are both absolutely integrable, then

(a) $g(t)$ is left sided and $h(t)$ is right sided

(b) $g(t)$ is right sided and $h(t)$ is left sided

(c) both $g(t)$ and $h(t)$ are right sided

(d) both $g(t)$ and $h(t)$ are left sided

Q.4 If $x_1(t) = e^{-2t}u(t)$ and $x_2(t) = e^{-3t}u(t)$ are related to $y(t)$ as $y(t) = x_1(t-2) * x_2(-t+3)$ then $Y(s)$ is

(a) $\dfrac{e^{-5s}}{(s+2)(s-3)}$ (b) $\dfrac{e^{-5s}}{(s+2)(3-s)}$

(c) $\dfrac{e^{s}}{(s+2)(s+3)}$ (d) $\dfrac{e^{+5s}}{(s+2)(s+3)}$

Q.5 A causal LTI system with impulse response $h(t)$ has following properties

1. Output is $y(t) = \dfrac{1}{6}e^{2t}$ for all t when input is $x(t) = e^{2t}$
2. The $h(t)$ satisfy following differential equation $\dfrac{d}{dt}h(t) + 2h(t) = e^{-4t}u(t) + ku(t)$, where '$k$' is constant

The value of 'k' will be

(a) 0 (b) 1

(c) 6 (d) 2

Q.6 The inverse Laplace transform of $\dfrac{(s+1)^2}{s^2 - s + 1}$ is

(a) $\delta(t) - \left[3e^{-t/2}\cos\dfrac{\sqrt{3}}{2}t + \sqrt{3}e^{-t/2}\sin\dfrac{\sqrt{3}}{2}t\right]u(t)$

(b) $\delta(t) + \left[3e^{-t/2}\cos\dfrac{\sqrt{3}}{2}t + 3e^{-t/2}\sin\dfrac{\sqrt{3}}{2}t\right]u(t)$

(c) $\delta(t) + \left[3e^{-t/2}\cos\dfrac{\sqrt{3}}{2}t + \sqrt{3}e^{-t/2}\sin\dfrac{\sqrt{3}}{2}t\right]u(t)$

(d) $\delta(t) + \left[\sqrt{3}e^{-t/2}\cos\dfrac{\sqrt{3}}{2}t + 3e^{-t/2}\sin\dfrac{\sqrt{3}}{2}t\right]u(t)$

Q.7 The Laplace transform of $x(t) = \delta(t^2 - 3t + 2)$ is

(a) e^{-3s} (b) $e^{-s} + e^{-2s}$

(c) $e^{-s} - e^{-2s}$ (d) $-e^{-s} - e^{-2s}$

CHAPTER 7

z-Transform

Introduction

The discrete-time counterpart of Laplace transform is z-transform. The frequency domain analysis of discrete-time system allows us to represent any arbitrary signal $x[n]$ as a sum of exponentials of the form z^n.

There are two varieties of z-transform: bilateral and unilateral. The bilateral one, also known as two- sided z-transform can handle all causal and non causal signals. It provides insights about system's characteristics such as stability, causality and frequency response.

The unilateral one, also known as one-sided z-transform can handle only causal signals and mainly used to solve difference equations with initial conditions.

7.1 The Definition

Consider a discrete-time signal $x[n]$. Its z-transform is defined as

$$ZT\{x(n)\} = X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

Where 'z' is complex variable which is given as

$$z = re^{j\Omega}$$

Where, r = distance from origin in z-plane (magnitude of z)

Ω = an angle from positive real axis in z-plane (angle of z)

The signal $x[n]$ and $X[z]$ makes z-transform pair as

$$x[n] \xleftrightarrow{ZT} X(z)$$

and the inverse z-transform which gives $x[n]$ back from $X(z)$ can be obtained as

$$ZT^{-1}\{X(z)\} = \text{IZT}\{X(z)\} = x[n] = \frac{1}{2\pi j} \cdot \oint X(z)z^{n-1}dz$$

The symbol $\oint$ indicates an integration in counter clockwise direction around a closed path in the complex plane.

7.2 Region of Covergence for z-transform

The z-transform is guarnteed to converge if $x[n] \cdot r^{-n}$ is absolutely summable.

hence $$\sum \left| x[n] \cdot r^{-n} \right| < \infty$$

This condition shows the $X(z)$ will be finite

$$|X(z)| < \infty$$

Thus, ROC consist of those values of $|z| = \left| re^{j\Omega} \right| = r$ for which z-transform converges.

7.2.1 Properties of ROC

(i) The ROC of $X(z)$ consists of a ring in the z-plane centred about the origin.

(ii) The ROC does not contain any poles.

(iii) If $x[n]$ is of finite duration, then the ROC is entire z-plane, except possibly $z = 0$ and/or $z = \infty$.

(iv) If $x[n]$ is a right-sided sequence, and if the circle $|z| = r_0$ is in the ROC then all finite values of z for which $|z| > r_0$ will also be in the ROC.

(v) If $x[n]$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also be in the ROC.

(vi) If x[n] is two sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$.

(vii) If the z-transform $X(z)$ of $x[n]$ is rational, then its ROC is bounded by poles or extends to infinity.

(viii) If the z-transform $X(z)$ of $x[n]$ is rational and if $x[n]$ is right sided, then the ROC is the region in the z-plane outside the outer most pole i.e., outside the circle of radius equal to the largest magnitude of the poles of $X(z)$. Furthermore, if $x[n]$ is causal (i.e., if it is right sided and equal to 0 for $n < 0$), then the ROC also includes $z = \infty$.

(ix) If the z-transform $X(z)$ of $x[n]$ is rational, and if $x[n]$ is left sided, then the ROC is the region in the z-plane inside the innermost non-zero pole i.e. magnitude of the poles of $X(z)$ other than any at $z = 0$ and extending inward to and possibly including $z = 0$. In particular, if $x[n]$ is anticausal (i.e., if it is left sided and equal to 0 for $n > 0$), then the ROC also includes $z = 0$.

7.2.2 The z-plane and poles and zeros

The graphical representation of complex number z = $re^{j\Omega}$ in terms of complex plane is called as z-plane

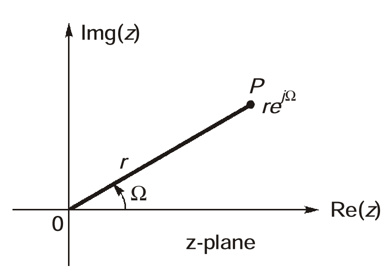


Figure-7.1 : *z-plane*

As we have seen z-transform reduces to DTFT on the contour in z-plane with $r = 1$, Thus

$$X(z)\big|_{z=e^{j\Omega}} = X(\Omega)$$

$z = e^{j\Omega}$ represents a unit circle z-plane as shown in the figure

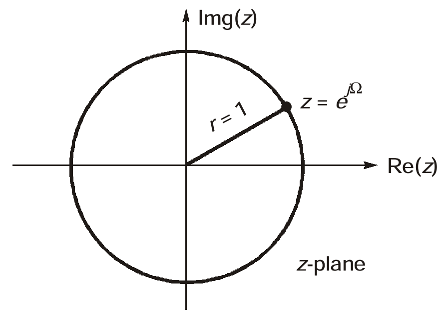


Figure-7.2

It is concluded that DTFT corresponds to z-transform evaluated on unit circle.

Poles and Zeros

Consider a ratio of two polynomials, say $F(z)$

$$F(z) = \frac{N(z)}{D(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + ...b_1 z + b_0}{a_n z^n + a_{n-1} z^{n-1} + ...a_1 z + a_0}$$

for above function, pole and zeros are defined as

Zeros: Roots of $N(z) = 0$ are defined as zeros when function $N(z)$ vanishes. It is represented as z_i.

i.e. $$\lim_{z \to z_i} F(z) = 0$$

Poles: Roots of $D(z) = 0$ are defined as poles when function $F(z)$ becomes infinity. It is represented as p_i.

i.e. $$\lim_{z \to p_i} F(z) = \infty$$

Example-7.1 The ROC of z-transform of the discrete time sequence

$x(n) = \left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)$ is

(a) $\frac{1}{3} < |z| < \frac{1}{2}$

(b) $|z| > \frac{1}{2}$

(c) $|z| < \frac{1}{3}$

(d) $2 < |z| < 3$

Solution : (a)

$$x(n) = (1/3)^n u(n) - (1/2)^n u(-n-1)$$

$(1/3)^n u(n)$ is right sided signal so ROC will be

$$|z| > 1/3 \quad \text{... (i)}$$

$-(1/2)^n u(-n-1)$ is left sided signal so ROC will be

$$|z| < 1/2 \quad \text{... (ii)}$$

from (i) and (ii) we see that ROC of the function will be

$$1/3 < |z| < 1/2$$