Detailed Solutions

ESE-2025 Mains Test Series

E & T Engineering Test No : 4

Section A : Analog and Digital Communication + Control Systems

Q.1 (a) Solution:

In the signal flow graph shown in figure, there are two forward paths, five loops and two pairs of two nontouching loops.

The forward paths and the gains associated with them are as follows:

Forward path (x_1-x_2-x_3-x_4-x_5-x_6), $M_1 = (1)(G_1)(G_2)(G_3)(1) = G_1G_2G_3$

Forward path (x_1-x_2-x_4-x_5-x_6), $M_2 = (1)(G_4)(G_3)(1) = G_4G_3$

The loops and the gains associated with them are as follows:

Loop (x_2-x_3-x_2), $L_1 = (G_1)(-H_1) = -G_1H_1$

Loop (x_4-x_5-x_4), $L_2 = (G_3)(-H_2) = -G_3H_2$

Loop (x_2-x_3-x_4-x_5-x_2), $L_3 = (G_1)(G_2)(G_3)(-H_3) = -G_1G_2G_3H_3$

Loop (x_2-x_4-x_5-x_2), $L_4 = (G_4)(G_3)(-H_3) = -G_4G_3H_3$

Loop (x_5-x_5), $L_5 = -H_4$

The pairs of two nontouching loops and the products of gains associated with them are as follows:

Loop (x_2-x_3-x_2) and (x_4-x_5-x_4), $L_{12} = (-G_1H_1)(-G_3H_2) = G_1G_3H_1H_2$

Loop (x_2-x_3-x_2) and (x_5-x_5), $L_{15} = (-G_1H_1)(-H_4) = G_1H_1H_4$

Since all the loops are touching both the forward paths, thus $\Delta_1 = 1$ and $\Delta_2 = 1$.

The determinant of the signal flow graph is

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5) + (L_{12} + L_{15})$$

$$\Delta = 1 - (-G_1H_1 - G_3H_2 - G_1G_2G_3H_3 - G_4G_3H_3 - H_4) + (G_1G_3H_1H_2 + G_1H_1H_4)$$

$$= 1 + G_1H_1 + G_3H_2 + G_1G_2G_3H_3 + G_4G_3H_3 + H_4 + G_1G_3H_1H_2 + G_1H_1H_4$$

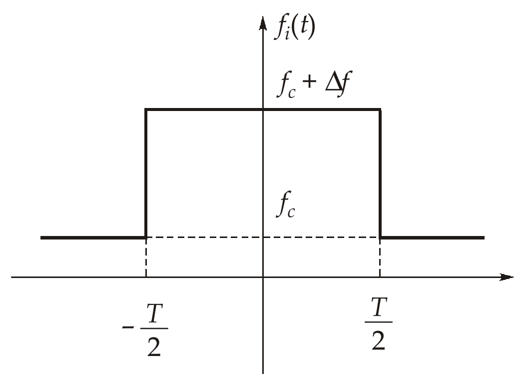
Applying Mason's gain formula, the transfer function is

$$\frac{x_6}{x_1} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta}$$

$$= \frac{G_1G_2G_3 + G_4G_3}{1 + G_1H_1 + G_3H_2 + G_1G_2G_3H_3 + G_4G_3H_3 + H_4 + G_1G_3H_1H_2 + G_1H_1H_4}$$

Q.1 (b) Solution:

The instantaneous frequency of the modulated wave $s(t)$ is as shown below:



We may thus express $s(t)$ as follows:

$$s(t) = \begin{cases} \cos(2\pi f_c t) & ; \; t < -\frac{T}{2} \\ \cos[2\pi(f_c + \Delta f)t] & ; -\frac{T}{2} \le t \le \frac{T}{2} \\ \cos(2\pi f_c t) & ; \; \frac{T}{2} < t \end{cases}$$

The Fourier transform of $s(t)$ is therefore,

$$S(f) = \int_{-\infty}^{-T/2} \cos(2\pi f_c t)e^{-j2\pi ft}dt + \int_{-T/2}^{T/2} \cos[2\pi(f_c + \Delta f)t]e^{-j2\pi ft}dt$$

$$+ \int_{T/2}^{\infty} \cos(2\pi f_c t)e^{-j2\pi ft}dt$$

$$= \int_{-\infty}^{\infty} \cos(2\pi f_c t) e^{-j2\pi ft} dt + \int_{-T/2}^{T/2} \{\cos[2\pi(f_c + \Delta f)t - \cos(2\pi f_c t)]\} e^{-j2\pi ft} dt$$

$$S(f) = \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c) + \frac{1}{2}\int_{-T/2}^{T/2} [e^{j2\pi(f_c+\Delta f - f)t} + e^{-j2\pi(f_c+\Delta f+f)t}]dt\right]$$

$$-\frac{1}{2}\int_{-T/2}^{T/2}\left[e^{j2\pi(f_c-f)t} + e^{-j2\pi(f_c+f)t}\right]dt$$

$$S(f) = \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)] + \frac{1}{2}\left[\frac{e^{j2\pi(f_c+\Delta f-f)t}}{j2\pi(f_c + \Delta f - f)} - \frac{e^{-j2\pi(f_c+\Delta f+f)t}}{j2\pi(f_c + \Delta f + f)}\right]_{-T/2}^{T/2}$$

$$-\frac{1}{2}\left[\frac{e^{j2\pi(f_c-f)t}}{j2\pi(f_c - f)} - \frac{e^{-j2\pi(f_c+f)t}}{j2\pi(f_c + f)}\right]_{-T/2}^{T/2}$$

$$S(f) = \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)] + \frac{1}{2}\left[\frac{\sin(\pi f_c + \Delta f - f)T)}{\pi(f_c + \Delta f - f)} - \frac{\sin(\pi(f_c + \Delta f + f)T)}{\pi(f_c + \Delta f + f)}\right]$$

$$-\frac{1}{2}\left[\frac{\sin(\pi(f_c + \Delta f - f)T)}{\pi(f_c + \Delta f - f)} - \frac{\sin(\pi(f_c + \Delta f + f)T)}{\pi(f_c + \Delta f + f)}\right]$$

$$S(f) = \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)] + \frac{T}{2}[\text{sinc}(f_c + \Delta f - f)T - \text{sinc}(f_c + \Delta f + f)T]$$

$$-\frac{T}{2}[\text{sinc}(f_c - f)T - \text{sinc}(f_c + f)T]$$

Q.1 (c) Solution:

The peak overshoot M_p is given by

$$M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$$

Taking natural logarithm on both sides,

$$\ln M_p = \frac{-\pi\xi}{\sqrt{1-\xi^2}}$$

On cross multiplying, we get

$$(1 - \xi^2)(\ln M_p)^2 = \pi^2\xi^2$$

i.e. $$\xi^2[\pi^2 + (\ln M_p)^2] = (\ln M_p)^2$$

$\therefore$ $$\xi^2 = \frac{[\ln M_p]^2}{\pi^2 + (\ln M_p)^2}$$

For $$M_p = 0.12$$

$$\ln M_p = -2.12$$

or $$(\ln M_p)^2 = 4.496$$

$\therefore$ $$\xi^2 = \frac{4.496}{9.869 + 4.496} = 0.313$$

$\therefore$ $$\xi = \sqrt{0.313} = 0.559$$

Therefore, resonant peak

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{1}{2 \times 0.559\sqrt{1-0.559^2}} = 1.079$$

Given, the peak time

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 0.2\,\text{sec}$$

$\therefore$ $$\omega_n = \frac{\pi}{t_p\sqrt{1-\xi^2}}$$

The undamped natural frequency,

$$\omega_n = \frac{\pi}{0.2\sqrt{1-0.559^2}} = 18.944 \text{ rad/s}$$

The resonant frequency

$$\omega_r = \omega_n\sqrt{1-2\xi^2} = 18.944\sqrt{1-2\times 0.559^2} = 11.6014 \text{ rad/s}$$

The bandwidth ω_b is given by

$$\omega_b = \omega_n\sqrt{(1-2\xi^2) + \sqrt{2-4\xi^2+4\xi^4}}$$

$$= 18.944\sqrt{(1-2\times 0.559^2) + \sqrt{2-4\times 0.559^2 + 4\times 0.559^4}}$$

$$= 22.76 \text{ rad/s}$$

The result is $M_r = 1.079$, $\omega_r = 11.6014$ rad/s

and $BW = 22.76$ rad/s

Q.1 (d) Solution:

(i) For binary phase shift keying,

The error probability is

$$P_e = Q\left[\sqrt{\frac{2E_b}{N_0}}\right] \quad ...(i)$$

Given,
$$\frac{N_0}{2} = 10^{-10} \text{ W/Hz}$$

$$N_0 = 2 \times 10^{-10} \text{ W/Hz}$$

$$\text{Pulse Duration, } T = \frac{1}{R} = \frac{1}{10^4} = 10^{-4} \text{ sec}$$

and
$$Q[4.74] = 10^{-6} = P_e \quad ...(ii)$$

From equation (i) and (ii),

$$4.74 = \sqrt{\frac{2E_b}{N_0}}$$

$$E_b = \frac{(4.74)^2 N_0}{2} = \frac{(4.74)^2 \times 2 \times 10^{-10}}{2}$$

$$= 2.24676 \times 10^{-9} \text{ J}$$

Also
$$E_b = \frac{A^2 T}{2} = 2.24676 \times 10^{-9} \quad ...(iii)$$

$$A^2 = \frac{2 \times 2.24676 \times 10^{-9}}{10^{-4}} = 4.49352 \times 10^{-5}$$

$$A = 6.7034 \times 10^{-3}$$

(ii) For data rate of 1 Mbps, $T = \frac{1}{10^6} = 10^{-6}$ sec

From equation (iii),

$$\frac{A^2 T}{2} = 2.24676 \times 10^{-9}$$

$$A^2 = \frac{2 \times 2.24676 \times 10^{-9}}{10^{-6}}$$

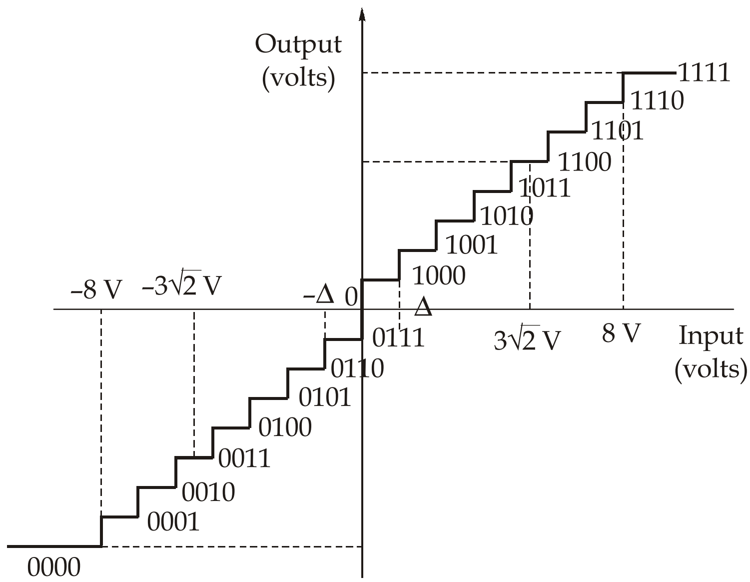
$$A = 67.033 \times 10^{-3}$$

Q.1 (e) Solution:

(i) The given signal is

$$m(t) = 6\sin(2\pi t) \text{ volts}$$

The input-output characteristics of a midrise type 4-bit quantizer (2^4 = 16 steps) with a step size of 1 V is as given below,



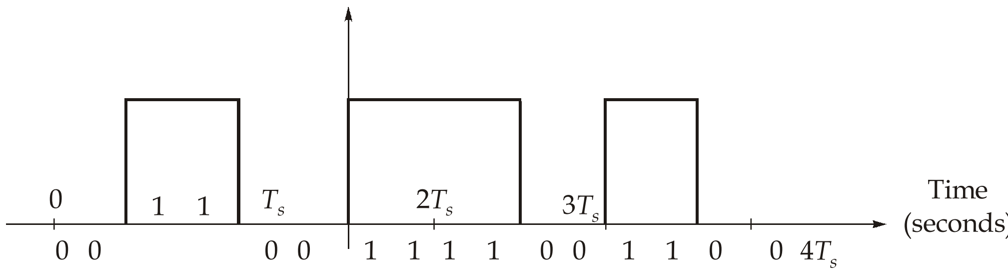
The time-period of $m(t)$ is given by the sampling instants is,

$$T = \frac{2\pi}{2\pi} = 1\,\text{sec}$$

Over one complete cycle of input i.e. -0.5 sec $\le t \le 0.5$ sec, the output of the quantizer at

t	$m(t)$	Code
$-\frac{3}{8}$	$-3\sqrt{2}$ V	0011
$-\frac{1}{8}$	$-3\sqrt{2}$ V	0011
$\frac{1}{8}$	$3\sqrt{2}$ V	1100
$\frac{3}{8}$	$3\sqrt{2}$ V	1100

Assuming ON-OFF signalling, the resulting PCM wave can be drawn as below,



(ii) Given data is

$$\text{Bit rate} = 64 \text{ Mb/sec} = R_b$$

$$\text{Number of bits} = n = 12$$

Considering f_s as the sampling rate, we have

$$R_b = nf_s$$

$$64 \times 10^6 = 12f_s \quad \Rightarrow \quad f_s = \frac{64}{12}\times 10^6 = \frac{16}{3}\text{MHz}$$

According to sampling theorem, for perfect reconstruction of signal,

$$f_s \geq 2f_m$$

$$\frac{16}{3} \geq 2f_m$$

$$f_m \leq \frac{8}{3}\text{MHz}$$

Hence, the maximum bandwidth of $m(t)$ for satisfactory operation is 2.67 MHz.

$$\text{Signal power, } S = \frac{A_m^2}{2}$$

For a uniform quantizer,

$$\text{Noise power, } N_Q = \frac{\Delta^2}{12} \quad \{\text{where } \Delta = \text{step size} = \frac{2A_m}{2^n}\}$$

$$\therefore \quad \frac{S}{N_Q} = \frac{A_m^2 (2^n)^2 \times 12}{2(2A_m)^2} = \frac{3}{2}(2^{2n}) = \frac{3}{2}(2^{24}) = 2.516\times 10^7$$

$$\left(\frac{S}{N_Q}\right)(\text{in dB}) = 10\log_{10}(2.516 \times 10^7) = 74 \text{ dB}$$

Q.2 (a) Solution:

From the provided information, the system has two poles at $s = 0$ (Type 2 system), one pole at $s = -5$ and one zero at $s = -A$. Thus, we can write open loop transfer function of the system as,

$$G(s) = \frac{k(s+z)}{s^2(s+5)}$$

For the input $R(s) = \frac{3}{s^3}$, on taking inverse Laplace transform, we get

$$r(t) = \frac{3}{2}t^2 \quad \text{i.e., parabolic input } r(t) = \frac{At^2}{2}$$

For parabolic input,

$$k_a = \lim_{s \to 0} s^2 G(s) = \lim_{s \to 0} s^2 \times \frac{k(s+z)}{s^2(s+5)} = \frac{kz}{5} \quad \text{...(i)}$$

$$\text{steady state error, } e_{ss} = \frac{A}{k_a} = 0.2$$

$$= \frac{3 \times 5}{kz} = 0.2$$

$$\Rightarrow \quad kz = 75 \quad \text{(ii)}$$

We have, $$|G(j\omega)| = \frac{k\sqrt{\omega^2 + z^2}}{\omega^2\sqrt{\omega^2 + 25}}$$

At $\omega = 1$ rad/sec, $|G(j1)| = 10^{24.94/20} = 17.66$

Thus,

$$17.66 = \frac{k\sqrt{1+z^2}}{\sqrt{26}}$$

Squaring both sides, we get

$$8108 = k^2 + k^2z^2$$

Using equation (ii),

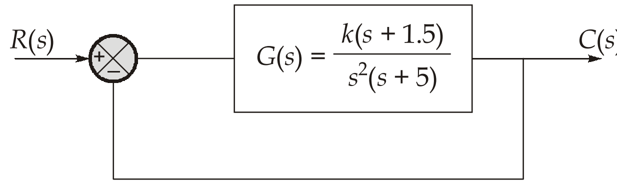
$$k^2 = 8108 - 5625$$

$$k^2 = 2483$$

$$\Rightarrow \quad k \approx 50$$

From equation (ii), $$z = \frac{75}{50} = 1.5$$

The unity feedback system is shown as below,



The closed loop transfer function of the feedback system is given by

$$\frac{C(s)}{R(s)} = \frac{k(s+1.5)}{s^2(s+5)+k(s+1.5)}$$

$$\frac{C(s)}{R(s)} = \frac{k(s+1.5)}{s^3+5s^2+ks+1.5k}$$

For $$r(t) = \frac{3}{2}t^2 \quad \text{or} \quad R(s) = \frac{3}{s^3}.$$

We get output as $$C(s) = \frac{k(s+1.5)}{(s^3+5s^2ks+1.5k)}\times\frac{3}{s^3}$$

$$C(s) = \frac{3k(s+1.5)}{s^3(s^3+5s^2+ks+1.5k)}$$

Q.2 (b) Solution:

(i) It is given that, the time gap between first peak overshoot and first peak undershoot is 0.785 sec i.e.,

$$T = \frac{2\pi}{\omega_d}-\frac{\pi}{\omega_d} = 0.785 \Rightarrow \frac{\pi}{\omega_d} = 0.785 \qquad \text{...(i)}$$

Since, time taken for peak undershoot/peak overshoot is given by

$$T = \frac{n\pi}{\omega_d} \quad \text{where } n = 1, 3, 5 \text{ ..... for peak overshoot}$$

$$n = 2, 4, 6 \text{ ..... for peak undershoot}$$

Settling time for 5% tolerance band,

$$t_s = \frac{3}{\xi\omega_n}$$

Since the number of cycles completed before the output is settled within 5% of its final value is 0.6366 cycles, thus

$$\frac{3}{\xi\omega_n T} = 0.6366$$

where, $T = \dfrac{2\pi}{\omega_d}$

$$\Rightarrow \quad \frac{3\times\omega_d}{\xi\omega_n\times 2\pi} = 0.6366$$

As, $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\Rightarrow \quad \frac{3\sqrt{1-\xi^2}}{\xi\times 2\pi} = 0.6366$$

$$\frac{0.477\sqrt{1-\xi^2}}{\xi} = 0.6366$$

or, $$\frac{\sqrt{1-\xi^2}}{\xi} = 1.3346$$

$$\frac{1-\xi^2}{\xi^2} = 1.78$$

$$1 - \xi^2 = 1.78\xi^2$$

$$2.78\xi^2 = 1$$

$$\xi^2 \simeq 0.36$$

$$\xi = 0.6$$

Now, from (i)

$$\frac{\pi}{\omega_d} = 0.785 \qquad \text{where} \quad \omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\Rightarrow \quad \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 0.785$$

$$\frac{\pi}{0.785\sqrt{1-\xi^2}} = \omega_n$$

$$\omega_n = \frac{\pi}{0.785\sqrt{1-(0.6)^2}}$$

$$\omega_n = 5 \text{ rad/sec}$$

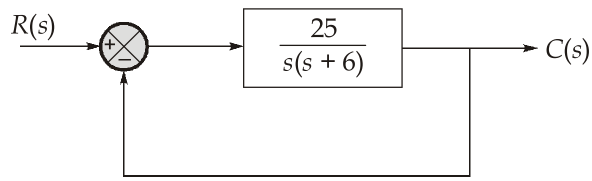
Thus, the transfer function of the required second order system can be written as

$$T(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{(5)^2}{s^2+2(0.6)(5)s+(5)^2}$$

$$T(s) = \frac{25}{s^2+6s+25} = \frac{G(s)}{1+G(s)},$$

where $G(s)$ is the open loop transfer function of the unity feedback system.

Thus, the system can be designed as below,



(ii) For the given system,

$$T(s) = \frac{C(s)}{R(s)} = \frac{\frac{25K}{s(s+5)}}{1+\frac{25K}{s(s+5)}} = \frac{25K}{s^2+5s+25K} = \frac{1}{1+\frac{s^2+5s}{25K}}$$

Therefore $$S_K^T = \frac{\partial T/T}{\partial K/K} = \frac{\partial T}{\partial K}\times\frac{K}{T}$$

$$= \frac{25(s^2+5s+25K)-625K}{(s^2+5s+25K)^2}\times\frac{s^2+5s+25K}{25}$$

$$= \frac{s(s+5)}{s^2+5s+25K}$$

Since the normal value of K is 1, we have

$$S_K^T = \frac{s(s+5)}{s^2+5s+25}$$

Q.2 (c) Solution:

(i) The DSB-SC modulated signal with carrier $c(t)$ and message signal $m(t)$ is given by,

$$s(t) = 5\cos(2\pi f_c t)m(t)$$

Taking the Fourier transform on both sides,

$$S(f) = \frac{5}{2}[M(f+f_c)+M(f-f_c)]$$

$$= 2.5[M(f+f_c)+M(f-f_c)]$$

The spectrum of message signal can be determined as follows:

$$\text{sinc}(t)\xleftrightarrow{CTFT}\text{rect}(f)$$

$$\text{sinc}^2(t)\xleftrightarrow{CTFT}\text{rect}(f)*\text{rect}(f)$$

Using the time-scaling property of Fourier Transform,

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

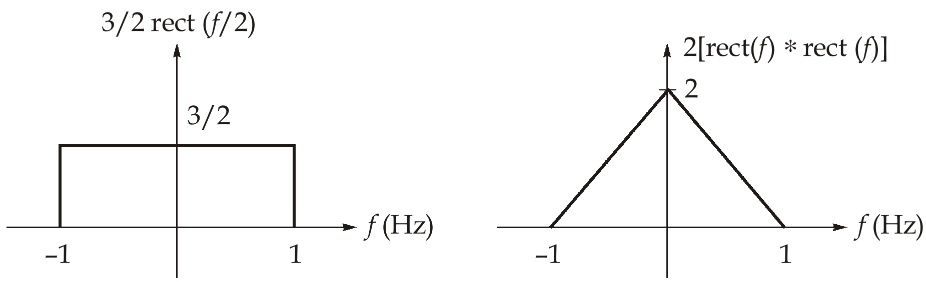
We get,

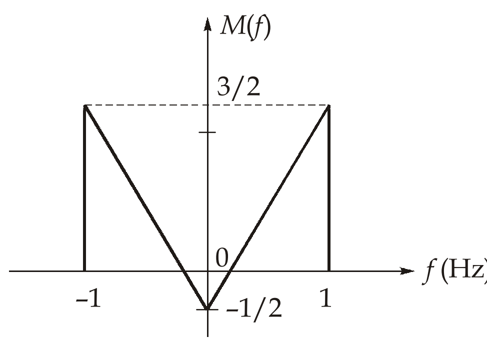
$$3\,\text{sinc}(2t) \xleftrightarrow{CTFT} \frac{3}{2}\text{rect}\left(\frac{f}{2}\right)$$

Given,

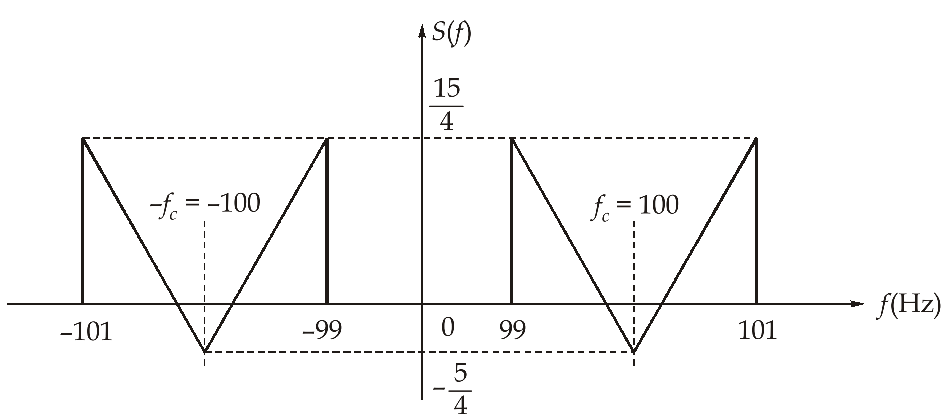
$$m(t) = 3\,\text{sinc}(2t) - 2\,\text{sinc}^2(t)$$

$$M(f) = \frac{3}{2} rect\left(\frac{f}{2}\right) - 2[\text{rect}(f) * \text{rect}(f)]$$

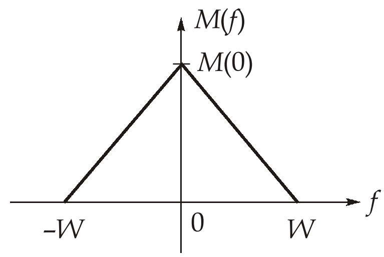




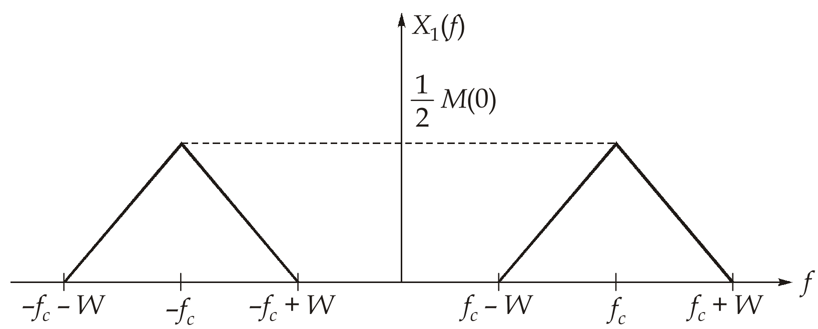
The spectrum of the DSB-SC modulated signal will be,



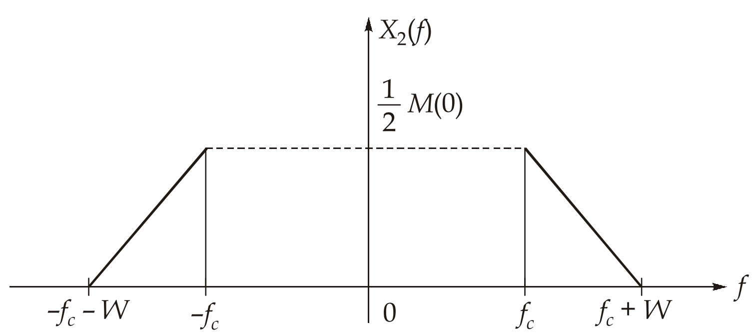
(ii) The spectrum of the signals at various points in the given system can be plotted as follows:



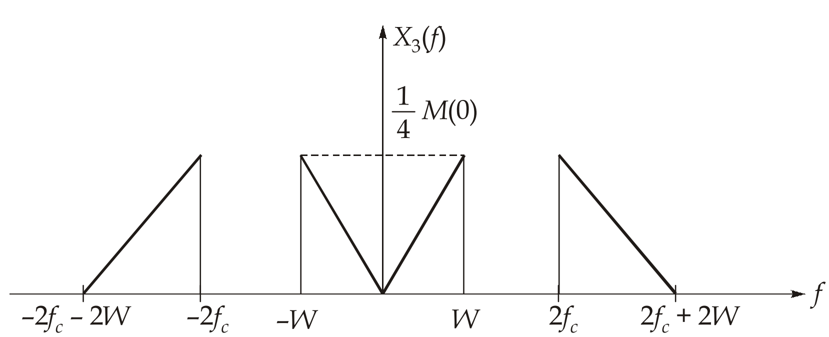
We have, $$X_1(f) = \frac{1}{2}[M(f-f_c)+M(f+f_c)]$$



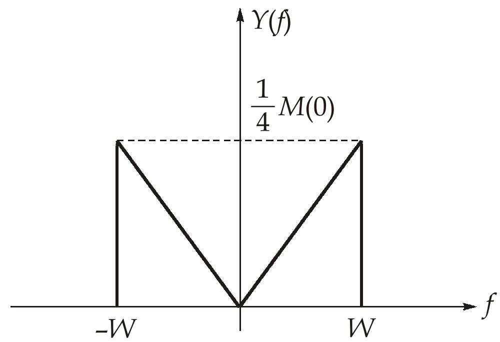
The spectrum of the signal after passing through the high-pass filter with cut-off frequency f_c,



We have, $$X_3(f) = \frac{1}{2}[X_2(f-f_c-W)+X_2(f+f_c+W)]$$



The spectrum of the signal after passing through the low-pass filter with cut-off frequency f_c is thus obtained as below,



Q.3 (a) Solution:

The steady state-error for ramp input to be zero, the system must be of type-2 i.e. two open-loop poles at the origin. Since one pole at s = 0 is already present in the given transfer function, thus the controller should have an integrator.

The closed loop system also has a zero. So, the controller will be proportional + integral (*PI*) controller.

Let the transfer function of the controller be,

$$G_c(s) = k_p + \frac{k_I}{s}$$

The forward-path transfer function of the controlled unity negative feedback system is

$$G(s) = G_p(s) \times G_c(s)$$

$$= \frac{4}{s(s+9)} \cdot \left(k_p + \frac{k_I}{s}\right)$$

$$G(s) = \frac{4(k_p s + k_I)}{s^2(s+9)}$$

The closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$= \frac{\dfrac{4(k_p s + k_I)}{s^2(s+9)}}{1 + \dfrac{4(k_p s + k_I)}{s^2(s+9)}} = \frac{4(k_p s + k_I)}{s^2(s+9) + 4(k_p s + k_I)}$$

$$\frac{C(s)}{R(s)} = \frac{4(k_p s + k_I)}{s^3 + 9s^2 + 4k_p s + 4k_I}$$

Given that the closed-loop system has a zero at $s = -3$, thus

$$k_p s + k_I = 0$$

i.e., $$k_p(-3) + k_I = 0$$

$\therefore$ $$k_I = 3k_p \quad ...(1)$$

The characteristic equation of the controlled system is,

$$s^3 + 9s^2 + 4k_p s + 4k_I = 0 \quad ...(2)$$

The system has complex poles which occur in conjugate pair, and let the third pole be at $s = -\alpha$

The characteristic equation can thus be written as

$$(s + \alpha)(s^2 + 2\xi\omega_n s + \omega_n^2) = 0$$

or, $$s^3 + (2\xi\omega_n + \alpha)s^2 + (\omega_n^2 + 2\xi\omega_n\alpha)s + \alpha\omega_n^2 = 0 \quad ...(3)$$

On comparing equation (2) and (3),

$$2\xi\omega_n + \alpha = 9 \quad ...(4)$$

and $$\omega_n^2 + 2\xi\omega_n\alpha = 4k_p \quad ...(5)$$

$$\alpha\omega_n^2 = 4k_I = 12k_p \quad ...(6)$$

Given, $$\omega_n = 5 \text{ rad/sec.}$$

From equation (4),

$$10\xi + \alpha = 9$$

$$\xi = \frac{9-\alpha}{10} \quad ...(7)$$

From equation (5),

$$25 + 10\xi\alpha = 4k_p \quad ...(8)$$

From equation (6),

$$25\alpha = 12k_p$$

$$25\alpha = 3(25 + 10\xi\alpha) \quad \text{...using equation (8)}$$

$$25\alpha = 75 + 30\xi\alpha$$

$$25\alpha = 75 + 30\alpha\left(\frac{9-\alpha}{10}\right) \quad \text{...using equation (7)}$$

$$25\alpha = 75 + 27\alpha - 3\alpha^2$$

$$3\alpha^2 - 2\alpha - 75 = 0$$

On solving above quadratic equation,

$$\alpha = \frac{2 \pm \sqrt{904}}{6} = \frac{2 \pm 30.06}{6}$$

$$\alpha = 5.344, -4.677$$

Considering, $\alpha = 5.344$

$[\because \alpha = -4.677$ leads to $\xi > 1$ and thus, no complex poles$]$

$$\xi = \frac{9-\alpha}{10} = \frac{9-5.344}{10} = 0.3656$$

$\therefore$ $\xi = 0.3656$

$$k_p = \frac{25\alpha}{12} = \frac{25 \times 5.344}{12} = 11.133$$

$$k_I = 3k_p = 3 \times 11.133 = 33.4$$

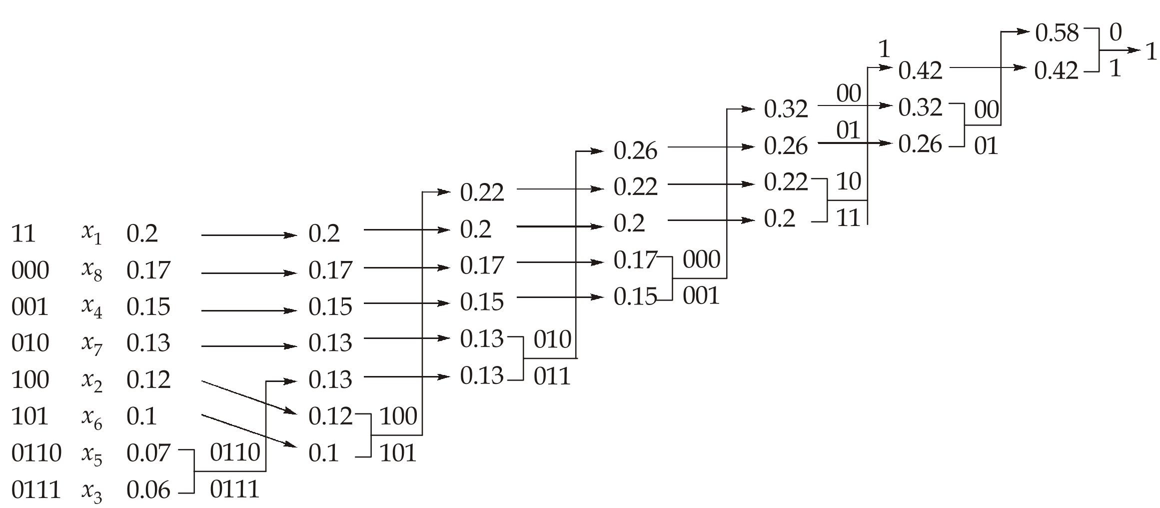
So, the controller will be

$$G_C(s) = k_p + \frac{k_I}{s} = 11.133 + \frac{33.4}{s}$$

$$G_C(s) = 11.133\left(1 + \frac{3}{s}\right)$$

Q.3 (b) Solution:

(i) Huffman Code:



From the above, we can write the Huffman code as below,

x_i	$P(x_i)$	code
x_1	0.2	11
x_2	0.12	100
x_3	0.06	0111
x_4	0.15	001
x_5	0.07	0110
x_6	0.1	101
x_7	0.13	010
x_8	0.17	000

Average bits of information i.e. the entropy of the source is given by,

$$H(x) = -\sum_{i=1}^{\infty} P(x_i)\log_2 P(x_i)$$

$$= -[0.2 \log_2 0.2 + 0.17 \log_2 0.17 + 0.15 \log_2 0.15 + 0.13 \log_2 0.13 + 0.12 \log_2 0.12 + 0.1 \log_2 0.1 + 0.07 \log_2 0.07 + 0.06 \log_2 0.06]$$

$$= 2.9035 \text{ bits/symbol}$$

The average codeword length is given by

$$H_{max} = \bar{L} = \sum_{i=1}^{\infty} L_i P(x_i)$$

$$= 0.2 \times 2 + 0.17 \times 3 + 0.15 \times 3 + 0.13 \times 3 + 0.12 \times 3 + 0.1 \times 3 + 0.07 \times 4 + 0.06 \times 4$$

$$= 2.93 \text{ bits/symbol}$$

Thus, efficiency,
$$\eta = \frac{H}{H_{max}} = \frac{2.9035}{2.93} \times 100 = 99.095\%$$

(ii) The capacity of the additive white Gaussian channel is given by the Shannon-Hartley Theorem as:

$$C = \frac{1}{2}\log_2\left(1 + \frac{P}{N_0 W}\right)$$

where W is the channel bandwidth, P is the average signal power received and N_0 is the one sided power spectral density of the AWGN.

For the non white Gaussian noise channel, although the noise power is equal to the noise power in the white Gaussian noise channel, but the noise power spectral density is not constant across all frequencies. For the non-white Gaussian Noise, since noise samples are correlated, knowledge of the previous noise samples provides partial information on the future noise samples and therefore, reduces their effective variance leading to a higher channel capacity.

Q.3 (c) Solution:

(i) Given,

Discrete time stochastic process $X(n) \equiv X(nT)$

$$\text{Sampling interval, } f_s = \frac{1}{T}$$

Assume, we will denote the discrete-time process by the subscript d and the continuous-time (analog) process by the subscript a. Also, f will denote the analog frequency and f_d the discrete-time frequency.

1.
$$\begin{aligned} R_d(k) &= E[X^*(n)\, X(n+k)] \\ &= E[X^*(nT)\, X(nT+kT)] \\ &= R_a(kT) \end{aligned}$$

Hence, the autocorrelation function of the sampled signal $X(n)$ is obtained by sampling the continuous-time autocorrelation function of $X(t)$ at integer multiples of the sampling interval T.

2.
$$\begin{aligned} R_d(k) &= R_a(kT) = \int_{-\infty}^{\infty} S_a(f) e^{j2\pi fkT}\, df \\ &= \sum_{l=-\infty}^{\infty} \int_{\frac{(2l-1)}{2T}}^{\frac{2l+1}{2T}} S_a(f) e^{j2\pi fkT}\, df \\ &= \sum_{l=-\infty}^{\infty} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} S_a\left(f+\frac{l}{T}\right) e^{j2\pi\left(f+\frac{l}{T}\right)kT}\, df \\ &= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \left[\sum_{l=-\infty}^{\infty} S_a\left(f+\frac{l}{T}\right)\right] e^{j2\pi fkT}\, df \end{aligned}$$

Let $f_d = fT$, then,

$$R_d(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{1}{T} \sum_{l=-\infty}^{\infty} S_a\left((f_d + l)/T\right) \right] e^{j2\pi f_d k} df_d \quad ...(1)$$

We know that the auto correlation function of a discrete-time process is the inverse Fourier transform of its power spectral density i.e.

$$R_d(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_d(f_d) e^{j2\pi f_d k} df_d \quad ...(2)$$

Comparing equation (1) and (2),

$$S_d(f_d) = \frac{1}{T} \sum_{l=-\infty}^{\infty} S_a\left(\frac{f_d + l}{T}\right) \quad ...(3)$$

3. From equation (3), we conclude that:

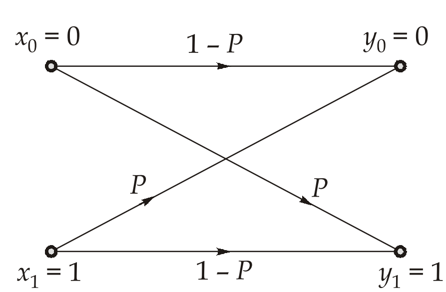
$$S_d(f_d) = \frac{1}{T} S_a\left(\frac{f_d}{T}\right)$$

only if X(t) is bandlimited. If $f_s = 1/T$ is the sampling frequency, then as per the Nyquist sampling rate, maximum frequency component of $X(t)$, $f_{max} = 1/2T$. Hence, the power spectral density of $X(n)$ is equal to the power spectral density of $X(t)$,

if: $$S_a(f) = 0, \forall f : |f| > \frac{1}{2T}$$

Otherwise, the sum of the shifted copies of S_a (in equation 3) will overlap and aliasing will occur.

(ii) Given:



1. The mutual information $I(x:y)$ for a binary symmetric channel is given by,

$$I(x:y) = H(Y) - H(Y/X)$$

We have, $$H(Y) = P(y_0)\log_2 \frac{1}{P(y_0)} + P(y_1)\log_2 \frac{1}{P(y_1)}$$

where $P(y_1) = P(y_1 | x_0) \cdot P(x_0) + P(y_1 | x_1) \cdot P(x_1) = PP_0 + (1 - P)(1 - P_0) = z$

and $P(y_0) = 1 - P(y_1) = 1 - z$

Thus, we have

$$H(Y) = H(z) = z\log_2 \frac{1}{z} + (1-z)\log_2 \frac{1}{1-z}$$

Also, we have

$$H(Y | X) = P(x = x_0)H(Y | X = x_0) + P(x = x_1)H(Y | X = x_1)$$
$$H(Y | X) = P_0H(P) + (1 - P_0)H(P) = H(P)$$

where, $$H(P) = P\log_2 \frac{1}{p} + (1-P)\log_2 \frac{1}{1-p}$$

Thus, we get

$$I(x:y) = H(Y) - H(Y | X) = H(z) - H(P)$$

2. The entropy $H(z)$ is dependent on P_0 and would be maximized when $z = (1 - z) = \frac{1}{2}$. Thus,

$$PP_0 + (1 - P)(1 - P_0) = \frac{1}{2}$$
$$2PP_0 - P_0 - P + 1 = \frac{1}{2}$$
$$P_0(2P - 1) = \frac{2P-1}{2}$$
$$P_0 = \frac{1}{2}$$

Hence, the value of P_0 that maximizes $I(x:y)$ is equal to $\frac{1}{2}$.

3. The channel capacity is given by

$$C = \max I(x:y) = \max\{H(z) - H(P)\}$$

From part (2), we know that $I(x:y)$ is maximized when $z = \frac{1}{2}$. Thus,

$$C = \frac{1}{2}\log_2 2 + \left(1 - \frac{1}{2}\right)\log_2 \frac{1}{1 - \frac{1}{2}} - H(P)$$

$$C = 1 - H(P)$$

Q.4 (a) Solution:

We have,

Open loop transfer function,

$$G(s) = \frac{K(s+2)}{(s+1)^2 + (\sqrt{2})^2}$$

or
$$G(s) = \frac{K(s+2)}{s^2 + 2s + 3}$$

- Open loop poles at $s = -1 + j\sqrt{2}$ and at $s = -1 - j\sqrt{2}$
- Open loop zeros at $s = -2$
- Centroid

$$C = \frac{\Sigma(\text{real part of poles}) - \Sigma(\text{real part of zeros})}{\text{Number of poles} - \text{Number of zeros}}$$

$$C = \frac{(-1-1)-(-2)}{2-1}$$

$$C = \frac{-2+2}{1} = 0$$

Coordinate of centroid, $C = (0, 0)$

- Angle of Asymptotes, $\theta = \frac{(2k+1)180°}{\text{number of poles} - \text{number of zeros}}$

where $k = 0, 1, \ldots$ upto (number of poles – number of zeros – 1)

Here, $k = 0$

Thus, the root locus plot has a single asymptote with

$$\theta = 180°$$

- **Break Away/Break in point:**

Characteristic equation,

$$1 + G(s) = 0$$

$$s^2 + 2s + 3 + K(s + 2) = 0$$

$$K(s + 2) = -(s^2 + 2s + 3)$$

$$K = \frac{-(s^2+2s+3)}{s+2}$$

At breakaway point,

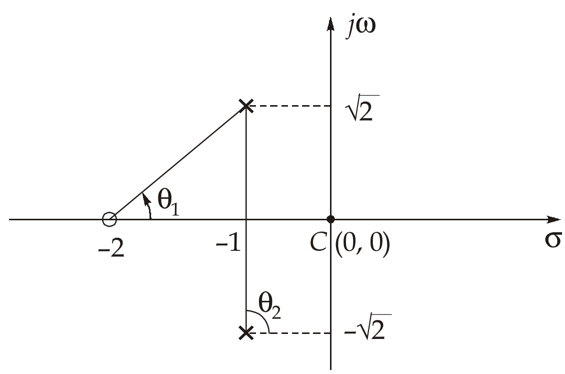
$$\frac{-dK}{ds} = \frac{(s+2)(2s+2)-(s^2+2s+3)}{(s+2)^2} = 0$$

$$\Rightarrow \quad 2s^2 + 2s + 4s + 4 - s^2 - 2s - 3 = 0$$

$$s^2 + 4s + 1 = 0$$

$$s = -0.27 \text{ and } s = -3.73$$

Plotting the open-loop poles and zeros,



The root locus exists on real axis to left of an odd number of poles and zeros of open loop transfer function, $G(s)H(s)$. Hence the root locus exists on the real axis for $\text{Re}(s) < -2$. Thus, root locus is not present on $s = -0.27$. Therefore, $s = -0.27$ is discarded as Break in/Break away point for root loci. Thus, $s = -3.73$ is a valid breakaway point.

- **Angle of departure for complex poles**

$$\theta_D = 180° - (\theta_2 - \theta_1),$$

where θ_2 is the sum of the angles from the other poles to the complex pole, and θ_1 is the sum of the angles from the zeros to the complex pole

For the complex pole $s = -1 + j\sqrt{2}$,

$$\theta_2 = 90° \quad \text{and} \quad \theta_1 = \tan^{-1}\sqrt{2}$$

$$\theta_{D1} = 180° - (90° - \tan^{-1}\sqrt{2})$$

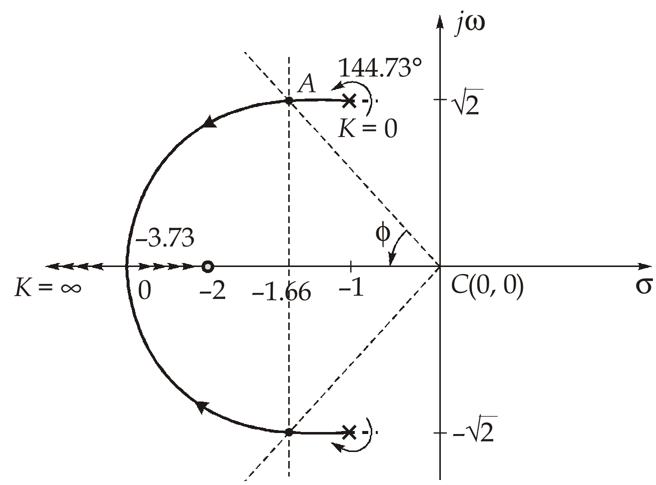
$$\theta_{D1} = 180° - 90° + 54.73°$$

$$\theta_{D1} = 144.73°$$

Similarly, for complex pole $s = -1 - j\sqrt{2}$

$$\theta_{D2} = -144.73°$$

Using the above information, the root loci of the given system can be drawn as below,



Point of root locus where $K = 1.33$

$$G(s) = \frac{K(s+2)}{s^2+2s+3} = \frac{1.33(s+2)}{s^2+2s+3}$$

For a point to exist on the root locus, the open-loop transfer function $G(s)$ at that point must satisfy the equation $G(s) = -1$. Thus,

$$\frac{1.33(s+2)}{s^2+2s+3} = -1$$

$$1.33s + 2.66 = -s^2 - 2s - 3$$

$$s^2 + 3.33s + 5.66 = 0$$

$$s = -1.66 + j1.7 \quad \text{or} \quad -1.66 - j1.7$$

And this point is represented by 'A' with $\phi = \tan^{-1}(1.7/1.66) = 45.68°$. Thus, damping ratio,

$$\xi = \cos\phi = \cos(45.68°) = 0.7$$

Q.4 (b) Solution:

(i) Given, $m(t) = A \tan h(\beta t)$

The condition required to eliminate the slope-overload distortion is,

$$\frac{\Delta}{T_s} \geq \left|\frac{dm(t)}{dt}\right|_{max},$$

where Δ is the step-size and T_s is the sampling interval.

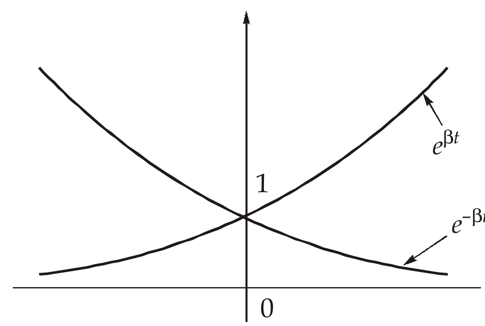
So, $$\Delta f_s \geq \frac{d}{dt}|A\tanh(\beta t)|_{\max}$$

$$\Delta f_s \geq |A\beta \,\text{sec}h^2(\beta t)|_{\max}$$

where $$\text{sec}h(\beta t) = \frac{1}{\cosh(\beta t)} = \frac{2}{e^{\beta t}+e^{-\beta t}}$$

Minimum value of $(e^{\beta t} + e^{-\beta t})$ occurs at $t = 0$ and the corresponding minimum value is 2.

So, $$|\text{sec}h^2(\beta t)|_{\max} = \left|\left(\frac{2}{2}\right)^2\right| = 1$$



Hence, $$\Delta f_s \geq A\beta$$

$$\Delta \geq \frac{A\beta}{f_s} = A\beta T_s$$

Thus, $$\Delta_{\min} = \frac{A\beta}{f_s} = A\beta T_s$$

(ii) The Nyquist sampling rate for a message signal is given by $2f_{m(\max)}$. Since, it is given that the signals are sampled at 25 percent above the Nyquist rate. Thus, the sampling rate of each message signal is,

$$f_{s1} = 1.25 \times 2 \times f_{m(\max)} = 1.25 \times 2 \times 12$$
$$= 30 \text{ kHz}$$

The net sampling rate of the TDM multiplexed signal is,

$$f_s = 30 \times 8 = 240 \text{ kHz}$$

The number of bits/sample can be determined as follows,

$$|Q_e|_{\max} \leq \left(\frac{0.6}{100}\right)m_p$$, where Q_e is the quantization error

$$\frac{\Delta}{2} \leq \left(\frac{0.6}{100}\right)m_p$$

Step-size, $$\Delta = \frac{m_p-(-m_p)}{2^n} = \frac{2m_p}{2^n};$$ n = number of bits/sample

So, $$\frac{m_p}{2^n} \leq \frac{3m_p}{500}$$

$$2^n \geq \frac{500}{3}$$

$$n \geq \log_2 \left\lceil \frac{500}{3} \right\rceil$$

$$n_{\min} = 8 \text{ bits/sample}$$

The minimum information rate is,

$$r_{\min} = n_{\min} f_s = 8 \times 240 = 1920 \text{ kbps}$$

Including the framing and synchronization bits, the minimum transmission data rate is

$$R_{b(\min)} = \left(1 + \frac{0.5}{100}\right) r_{\min} = \frac{201}{200} \times 1920$$

$$= 1929.6 \text{ kbps}$$

The minimum channel bandwidth required to transmit the TDM multiplexed signal is,

$$(\text{BW})_{\min} = \frac{R_{b(\min)}}{2} = \frac{1929.6}{2} = 964.8 \text{ kHz}$$

Q.4 (c) Solution:

(i) $A = \begin{bmatrix} 5 & 0 \\ 1 & -1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} : C^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

To check the controllability of the system:

$$Q_c = [B : AB] = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$$

$$|Q_c| = \begin{vmatrix} 1 & 5 \\ -1 & 2 \end{vmatrix} = 2 + 5 = 7 \neq 0$$

So, the given system is controllable

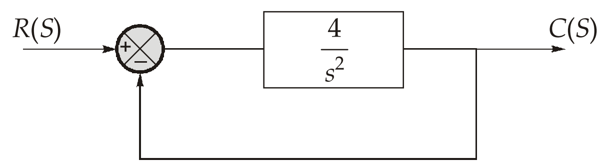
To check the absorbability of the system:

$$Q_0 = [C^T : A^T C^T] = \begin{bmatrix} 1 & 6 \\ 1 & -1 \end{bmatrix}$$

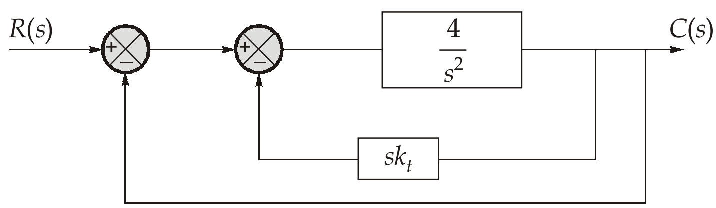
$$|Q_0| = -2 - 6 = -8 \neq 0$$

So the given system is observable.

(ii) Given uncompensated system,



With tachometer feedback, the block diagram of the compensated system is



The overall transfer function

$$\frac{C(s)}{R(s)} = \frac{\dfrac{\frac{4}{s^2}}{1+\frac{4}{s^2}(sk_t)}}{1+\dfrac{\frac{4}{s^2}}{1+\frac{4}{s^2}(sk_t)}} = \frac{\dfrac{4}{s^2+4sk_t}}{1+\dfrac{4}{s^2+4sk_t}} = \frac{4}{s^2+4sk_t+4}$$

The characteristic equation,

$$q(s) = s^2 + 4sk_t + 4$$

Comparing it with standard second order characteristic equation, we get.

$$\omega_n = \sqrt{4} \text{ rad/sec} = 2 \text{ rad/sec}$$

$$2\xi\omega_n = 4k_t$$

$$\xi = \frac{4k_t}{2\times 2} = k_t \qquad \text{...(i)}$$

Now given,

$$\% M_p = 50\%$$

$$0.5 = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}$$

$$-\frac{\xi\pi}{\sqrt{1-\xi^2}} = \ln(0.5)$$

$$\xi^2 = 0.3245$$

Substituting this value in equation (i),

$$2 \times 0.569 \times 2 = 4k_t$$

$$\therefore \quad k_t = 0.569$$

Section B : Analog and Digital Communication + Control Systems

Q.5 (a) Solution:

Given,

$$G(s) = \frac{(1+0.2s)(1+0.025s)}{s^3(1+0.005s)(1+0.001s)}$$

The sinusoidal transfer function is

$$G(j\omega) = \frac{(1+j\,0.2\omega)(1+j0.025\omega)}{(j\omega)^3(1+j0.005\omega)(1+j0.001\omega)}$$

Rationalizing,

$$G(j\omega) = \frac{(1+j0.2\omega)(1+j0.025\omega)(1-j0.005\omega)(1-j0.001\omega)}{-j\omega^3(1+j0.005\omega)(1-j0.005\omega)(1+j0.001\omega)(1-j0.001\omega)}$$

$$= \frac{\left(1-0.005\omega^2+j0.225\omega\right)\left(1-0.000005\omega^2-j0.006\omega\right)}{-j\omega^3\left(1+0.000025\omega^2\right)\left(1+0.000001\omega^2\right)}$$

$$= -\frac{0.225\left(1-0.000005\omega^2\right)-0.006\left(1-0.005\omega^2\right)}{\omega^2\left(1+0.000025\omega^2\right)\left(1+0.000001\omega^2\right)}$$

$$+\frac{j\left[\left(1-0.005\omega^2\right)\left(1-0.000005\omega^2\right)+0.00135\omega^2\right]}{\omega^3\left(1+0.000025\omega^2\right)\left(1+0.000001\omega^2\right)} \quad \text{...(i)}$$

When $\omega = 0$,

$$G(j0) = -\infty + j\infty = \infty\angle{-270^\circ}$$

When $\omega = \infty$,

$$G(j\infty) = -0 + j0 = 0\angle{-270^\circ}$$

For the plot to cross the real axis, the imaginary part of $G(j\omega)$ must be zero i.e.

$$(1 - 0.005\omega^2)(1 - 0.000005\omega^2) + 0.00135\omega^2 = 0$$

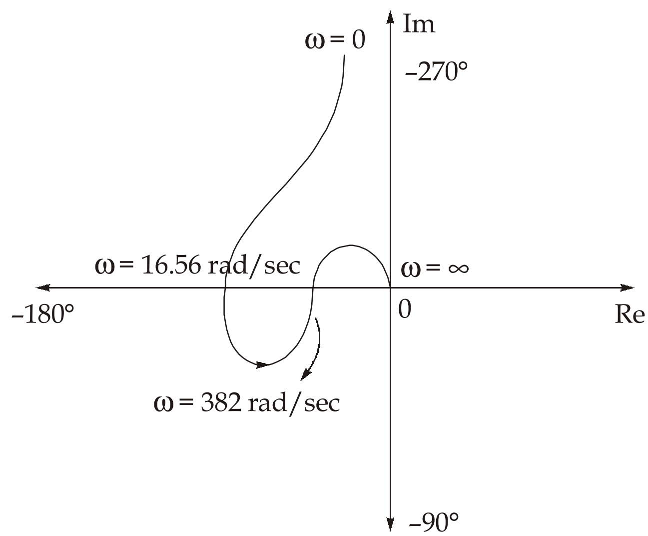
$$1 - 0.003655\omega^2 + 0.000000025\omega^4 = 0$$

$$\omega^4 - 146200\omega^2 + 40000000 = 0 \quad \Rightarrow \quad \omega^2 = 145926,\ 274.112$$

ω = 382 rad/sec, 16.56 rad/sec

The polar plot does not cross the real axis.

Based on the above information, an approximate polar plot is drawn as shown in figure.



Using equation (i),

$$G(j\omega)\big|_{\omega=382} = -5.7 \times 10^{-6}$$

$$G(j\omega)\big|_{\omega=16.56} = -8.22 \times 10^{-4}$$

Q.5 (b) Solution:

(i) Given data

frequency deviation, Δf = 10 kHz

Modulation frequency, f_m = 5 kHz

Two frequency multipliers are connected in cascade with,

$$n_1 = 2$$

$$n_2 = 3$$

The overall frequency multiplication ratio is

$$n = n_1 \times n_2 = 2 \times 3 = 6$$

Assume that the instantaneous frequency of the FM wave at the input of the first frequency multiplier is

$$f_{i1}(t) = f_c + \Delta f \cos(2\pi f_m t)$$

The instantaneous frequency of the resulting FM wave at the output of the second frequency multiplier is therefore,

$$f_{i2}(t) = nf_c + n\Delta f\cos(2\pi f_m t)$$
$$= f_c' + \Delta f'\cos(2\pi f_m t)$$

Thus, the frequency deviation of this FM wave is equal to

$$\Delta f' = n\Delta f = 6 \times 10 = 60 \text{ kHz}$$

In the FM signal, the adjacent sidebands are separated by message signal frequency f_m. At the output of second multiplier, the message signal frequency remains unchanged. Thus, the frequency separation of the adjacent sidebands of this FM wave is unchanged at

$$f_m = 5 \text{ kHz}$$

(ii) **1.** The bandwidth of AM signal is given by $2f_m$, where f_m is the frequency of the modulating signal.

Given, $\Delta f = 3 \times 2 f_m = 6 f_m$

Modulation index of FM signal,

$$m_f = \frac{\Delta f}{f_m} = \frac{6 f_m}{f_m} = 6$$

2. Let A_1 be the peak amplitude of the carrier in the AM system and A_2 be peak amplitude of the carrier in the FM system.

Total power in AM system $= \frac{A_1^2}{2} + \frac{1}{4}\mu^2 A_1^2$

where μ is the modulation index

Total power in FM system $= \frac{A_2^2}{2}$ [$\because$ $\beta > 1$, i.e., wide band FM signal]

The amplitude of n^{th} order sideband in FM signal is given by $A_c|J_n(\beta)|$. Thus, the magnitude of the first order sideband spaced at $\pm f_m$ Hz from carrier in FM system is

$$A_2|J_1(6)|$$

Magnitude of the sideband spaced at $\pm f_m$ Hz from carrier in AM system is $\frac{\mu}{2}A_1$

Since the total average powers are equal in both AM and FM systems, thus

$$\frac{A_2^2}{2} = \frac{A_1^2}{2} + \frac{\mu^2 A_1^2}{4} \quad \text{...(i)}$$

Since the magnitudes of sidebands spaced at $\pm f_m$ Hz from carriers in both systems are equal, thus

$$A_2|J_1(6)| = \frac{\mu}{2}A_1, \text{ where } J_1(6) = 0.34 \text{ (Given)} \quad \text{...(ii)}$$

Equations (i) and (ii) can be solved simultaneously to yield,

$$\mu = \frac{2|J_1(6)|}{\sqrt{1-2J_1^2(6)}} = \frac{2\times 0.34}{\sqrt{1-2\times(0.34)^2}} = 0.7755$$

Q.5 (c) Solution:

We have,

open loop zeros at $s = -2$ and at $s = -1$

open loop poles at $s = -0.1$ and at $s = 1$

Using the above information, we can write open loop transfer function of the system as

$$G(s) = \frac{K(s+1)(s+2)}{(s+0.1)(s-1)}$$

The characteristic equation of the system is given as

$$1 + G(s) = 0$$

$$(s + 0.1)(s - 1) + K(s + 1)(s + 2) = 0$$

$$(1 + K)s^2 + (3K - 0.9)s + (2K - 0.1) = 0$$

Applying Routh Hurwitz criteria, we get

s^2	$(1+K)$	$(2K-0.1)$
s^1	$(3K-0.9)$	
s^0	$(2K-0.1)$	

Now,

(i) No poles in right-half of s-plane (system is stable)

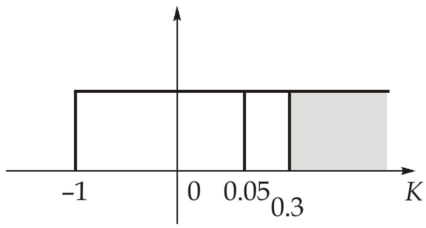
For getting all poles in left half of s-plane, all the elements of first column of Routh Hurwitz table must be of the same sign i.e.,

Case-I		**Case-II**
s^2 +	or	s^2 –
s^1 +		s^1 –
s^0 +		s^0 –
$\Downarrow$	or	$\Downarrow$
$1 + K > 0$		$1 + K < 0$
$K > -1$ ...(i)		$K < -1$ ...(i)
$3K - 0.9 > 0$		$3K - 0.9 < 0$
$K > 0.3$ ...(ii)		$K < 0.3$ ...(ii)

$2K - 0.1 > 0$ $\qquad$ $2K - 0.1 < 0$

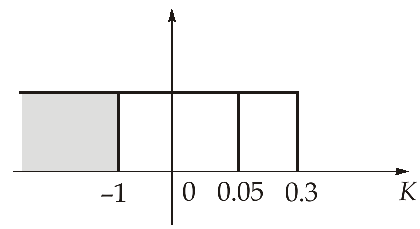
$K > 0.05 \quad ...(iii)$ $\qquad$ $K < 0.05 \quad ...(iii)$

On combining all the three conditions, we get



$K > 0.3$

or



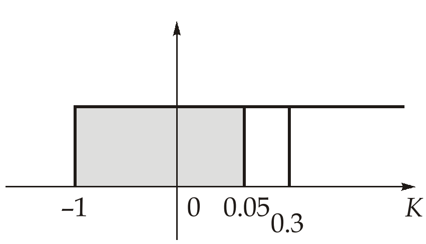
$K < -1$

We discard case-II because for negative feedback system, K must be positive. Therefore, for all the poles to lie in the left half of s-plane, K must be greater than 0.3.

(ii) 1 pole lie in the right half of s-plane.

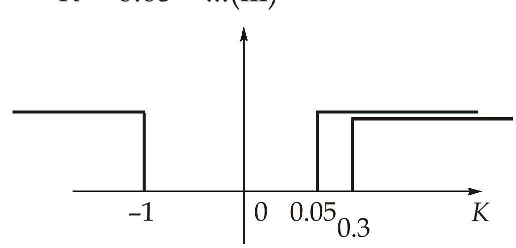
For 1-pole in the right half of s-plane, there should be only one sign change in first column of Routh Hurwitz table i.e.,

Case-I		**Case-II**
s^2 +	or	s^2 –
s^1 +		s^1 +
s^0 –		s^0 +
$\Downarrow$	or	$\Downarrow$
$1 + K > 0$		$1 + K < 0$
$K > -1 \quad ...(i)$		$K < -1 \quad ...(i)$
$3K - 0.9 > 0$		$3K - 0.9 > 0$
$K > 0.3 \quad ...(ii)$		$K > 0.3 \quad ...(ii)$
$2K - 0.1 < 0$		$2K - 0.1 > 0$
$K < 0.05 \quad ...(iii)$		$K > 0.05 \quad ...(iii)$



$-1 < K < 0.05$

or



No common region 0 is present.
Thus, case-II is discarded.

As K can't be negative for negative feedback system. Hence, for one pole to lie in right half of s-plane, $0 < K < 0.05$.

(iii) 2 Poles in right half of s-plane.

For 2 poles to lie in right half of s-plane, there must be two sign change in the first column of Routh-Hurwitz table

s^2 +

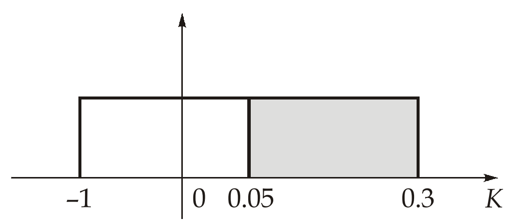
s^1 –

s^0 +

$1 + K > 0$; $\quad 3K - 0.9 < 0$; $\quad 2K - 0.1 > 0$

$K > -1$ ...(i) $\quad K < 0.3$ ...(ii) $\quad K > 0.05$ ...(iii)

On combining, we get



$0.05 < K < 0.3$

For no root to lie in left half of s-plane; K must be in between 0.05 and 0.3.

Q.5 (d) Solution:

Given data:

$$\text{Bit rate: } R_b = 4.5 \times 10^6 \text{ bits per second}$$

$$T_b = \frac{1}{R_b}$$

$$T_b = \frac{1}{4.5\times10^6}\text{sec} = \frac{2}{9}\times10^{-6}\text{ sec}$$

$$\text{Two sided PSD} = \frac{N_0}{2} = 10^{-20}\text{ W/Hz}$$

$$\text{Bit Energy, } E_b = T_b \times \text{Signal power} = T_b\times\frac{A^2}{2} = T_b\times\frac{(1.2\times10^{-6})^2}{2}$$

$$= \frac{2}{9}\times\frac{1.44\times10^{-12}}{2}\times10^{-6}$$

$$= 0.16 \times 10^{-18}\text{ Joule}$$

For digital signalling scheme,

Probability of error, $P_e = Q\left(\sqrt{\dfrac{d_{\min}^2}{2N_0}}\right)$

For FSK system, $d_{\min} = \sqrt{2E_b}$. Thus,

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = \frac{1}{2}erfc\left(\sqrt{\frac{E_b}{2N_0}}\right)$$

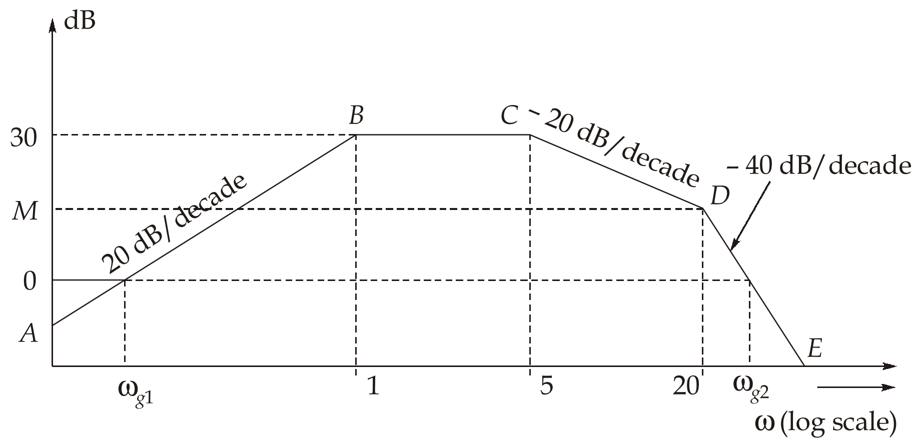
$$P_c = \frac{1}{2}erfc\left[\sqrt{\frac{0.16\times10^{-18}}{2\times2\times10^{-20}}}\right] = \frac{1}{2}erfc(2)$$

$$P_e = \frac{1}{2}\frac{e^{-4}}{\sqrt{\pi}\cdot 2} = 2.58\times10^{-3}$$

$$P_e = 0.026$$

Q.5 (e) Solution:

The bode plot of the minimum phase system is given as below,



(i) In a bode plot, a change in the slope by –20 dB/dec at ω_c indicates the presence of a pole at $s = -\omega_c$ and thus, the term $\dfrac{1}{1+\dfrac{s}{\omega_c}}$ in the transfer function. Similarly, a change in the slope by 20 dB/dec at ω_c indicates the presence of a zero at $s = -\omega_c$ and thus, the term $1+\dfrac{s}{\omega_c}$ in the transfer function. Also, an initial slope of 20 dB/sec indicates the presence of zero at $s = 0$. Thus, the transfer function of the system is given by

$$G(s) = \frac{ks}{(1+s)\left(1+\frac{s}{5}\right)\left(1+\frac{s}{20}\right)}$$

For $0 < \omega < 1$, the magnitude, $|G(j\omega)|$ is given by

$$|G(j\omega)| = 20 \log k + 20 \log \omega$$

At $\omega = 1$ rad/sec,

$$30 = 20 \times 1 \log 1 + 20 \log k$$

$$k = 10^{\frac{30}{20}} = 31.622$$

$$\therefore \qquad G(s) = \frac{31.622 s \times 5 \times 20}{(s+1)(s+5)(s+20)} = \frac{3162.2s}{(s+1)(s+5)(s+20)}$$

(ii) Two gain crossover frequencies,

The slope for line AB, $\omega_{g1} = ?$ $\omega_{g2} = ?$

$$20 \text{ dB/decade} = \frac{30-0}{\log 1 - \log \omega_{g1}}$$

$$\log \frac{1}{\omega_{g1}} = \frac{30}{20}$$

$$\omega_{g1} = 0.0316 \text{ rad/sec.}$$

The slope for line BC,

$$-20 \text{ dB/decade} = \frac{M-30}{\log \frac{20}{5}}$$

$$\Rightarrow \qquad M - 30 = -20 \log 4 = -12.04$$

$$M = 17.958 \text{ dB}$$

To calculate ω_{g2}, the slope for line CD,

$$-40 \text{ dB/decade} = \frac{0-M}{\log \omega_{g1} - \log 20} = \frac{0-17.958}{\log \frac{\omega_{g2}}{20}}$$

$$\log \frac{\omega_{g2}}{20} = 0.44895$$

$$\omega_{g2} = 56.234 \text{ rad/sec}$$

Q.6 (a) Solution:

(i) With $\alpha = 0$, the forward-path transfer function.

$$G(s) = \frac{16}{s(s+4)}$$

$$\text{Velocity error constant } k_v = \lim_{s\to 0} sG(s) = 4$$

$$e_{ss}\big|_{\text{unit ramp}} = \frac{1}{k_V} = 0.25$$

The characteristic equation of the system is

$$s^2 + 4s + 16 = 0$$

Comparison with the standard characteristic equation.

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

Gives $\omega_n = 4$ and $\xi = 0.5$

Therefore, $M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} \times 100 = 16.3\%$

(ii) With rate feedback, the forward-path transfer function is obtained by eliminating the inner loop:

$$G(s) = \frac{\frac{16}{s(s+4)}}{1+\frac{16\times s}{s(S+4)}} = \frac{16}{s(s+4+16\alpha)}$$

$$k_v = \frac{16}{4+16\alpha} = \frac{4}{1+4\alpha}$$

The characteristic equation now takes the form

$$s^2 + (4 + 16\alpha)s + 16 = 0$$

Corresponding to 1.5% overshoot, the damping ratio ξ is given by

$$\xi^2 = \frac{[\ln(0.015)]^2}{[\ln(0.015)]^2 + \pi^2}$$

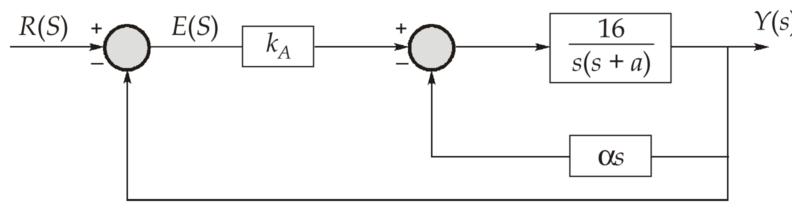
The value of α that results in $\xi = 0.8$ is given by the following equation

$$2 \times 0.8 \times 4 = 4 + 16\alpha$$

Solving, we get $\alpha = 0.15$

For this value of α, $e_{ss}\big|_{\text{unit ramp}} = \frac{1}{K_v} = \frac{1+4\alpha}{4} = 0.4$

(iii) The system of given figure may be extended by including an amplifier k_A as shown in figure. The forward-path transfer function now becomes.



$$G(s) = \frac{16k_A}{s(s+4+16\alpha)}$$

$$k_V = \frac{16k_A}{4+16\alpha} = \frac{4k_A}{1+4\alpha}$$

The required k_V = 4, therefore,

$$\frac{4k_A}{1+4\alpha} = 4 \quad \text{...(i)}$$

The characteristic equation of the extended system is

$$s^2 + (4 + 16\alpha)s + 16k_A = 0 \quad \text{...(ii)}$$

The requirement of $\xi = 0.8$ gives rise to the following equation:

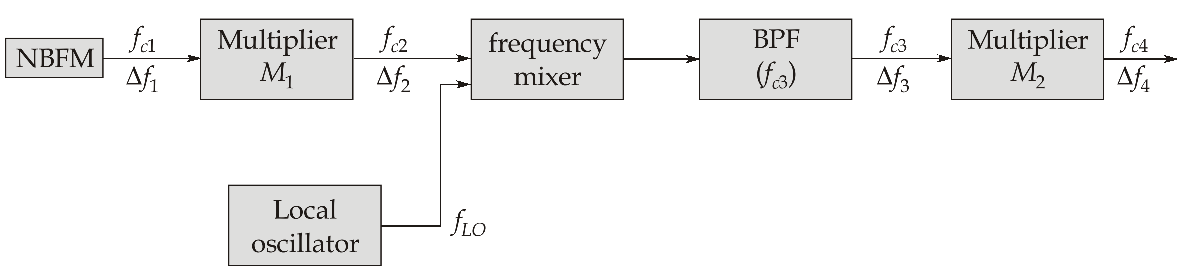
$$2\times0.8\times\sqrt{16k_A} = 4 + 16\alpha \quad \text{...(iii)}$$

From equation (i) and (iii), we obtain.

$$k_A = 2.56;\ \alpha = 0.39$$

Q.6 (b) Solution:

The modulator is shown in the figure below,



We need to determine M_1, M_2 and f_{LO}.

First the NBFM generator generates,

$$f_{c1} = 20000 \text{ Hz and } \Delta f_1 = 10 \text{ Hz}$$

The final WBFM should have

$$f_{c4} = 97.3 \times 10^6 \text{ Hz and } \Delta f_4 = 20480 \text{ Hz}$$

We first find the total factor of frequency multiplication needed as

$$M_1 \cdot M_2 = \frac{\Delta f_4}{\Delta f_1} = \frac{20480}{10} = 2^{11}$$

Because only frequency doublers can be used, we have three equations:

$$M_1 = 2^{n_1}$$

$$M_2 = 2^{n_2}$$

$$n_1 + n_2 = 11$$

It is also clear that

$$f_{c2} = 2^{n_1} f_{c1}$$

$$f_{c4} = 2^{n_2} f_{c3}$$

To find f_{LO}, there are three possible relationships:

$$f_{c3} = f_{c2} \pm f_{LO} \quad \text{and} \quad f_{c3} = f_{LO} - f_{c2}$$

Each should be tested to determine the one that will fall in

$$400 \text{ kHz} \le f_{LO} \le 500 \text{ kHz}$$

(i) First, Let $f_{c3} = f_{c2} - f_{LO}$. This case leads to

$$97.3 \times 10^6 = 2^{n_2} f_{c3}$$

$$= 2^{n_2}(f_{c2} - f_{LO})$$

$$= 2^{n_2}(2^{n_1} f_{c1} - f_{LO})$$

$$= 2^{n_2} 2^{n_1} f_{c1} - 2^{n_2} f_{LO}$$

$$f_{LO} = 2^{-n_2}(2^{11} \times 20 \times 10^3 - 9.73 \times 10^7)$$

$$f_{LO} = 2^{-n_2}(4.096 \times 10^7 - 9.73 \times 10^7) < 0$$

This is outside local oscillator frequency range.

(ii) Now, Let, $f_{c3} = f_{c2} + f_{LO}$. This case leads to

$$97.3 \times 10^6 = 2^{n_2}(2^{n_1} f_{c1} + f_{LO})$$

$$f_{LO} = 2^{-n_2}(9.73 \times 10^7 - 4.096 \times 10^7)$$

$$f_{LO} = 2^{-n_2}(5.634 \times 10^7)$$

If $n_2 = 7$ then $f_{LO} = 440$ kHz, which lies within the realizable range of the local oscillator.

(iii) Let,

$$f_{c3} = f_{LO} - f_{c2}$$

$$97.3 \times 10^6 = 2^{n_2}(f_{LO} - f_{c2})$$

$$= 2^{n_2} f_{LO} - 2^{n_2} \cdot 2^{n_1} f_{c1}$$

$$f_{LO} = 2^{-n_2}(9.73\times10^7 + 4.096\times10^7)$$

$$f_{LO} = 2^{-n_2}(13.826\times10^7)$$

For $n = 8$, f_{LO} = 540 kHZ and for $n = 9$, f_{LO} = 270 kHz.

Hence, no integer n_2 will lead to a realizable f_{LO}.

∴ Final design is,

$$M_1 = 2^{n_1} = 2^4 = 16$$

$$M_2 = 2^{n_2} = 2^7 = 128$$

$$f_{LO} = 440 \text{ kHz}$$

Q.6 (c) Solution:

(i) Given data is

$$m(t) = 3\cos(25\pi t) - 2\cos(50\pi t)$$

$$s(t) = 6\cos(2\pi f_c t) + m(t)\cos(2\pi f_c t), \qquad \text{where } f_c = 600 \text{ kHz}$$

$$= 6\left[1+\frac{1}{2}\cos(25\pi t)-\frac{1}{3}\cos(2\pi\times 25t)\right]\cos 2\pi f_c t$$

Comparing with the standard AM equation given by

$$s(t) = A_c\left[1+\mu_1\cos(2\pi f_{m1}t)+\mu_2\cos(2\pi f_{m2}t)\right]\cos(2\pi f_c t)$$

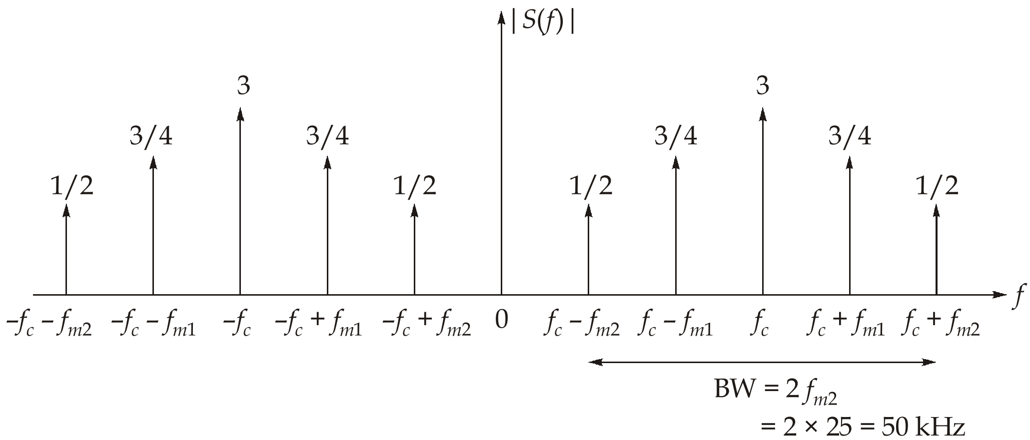
We get,

$A_c = 6,\ \mu_1 = \frac{1}{2},\ f_{m1} = 12.5\text{ kHz},\ \mu_2 = \frac{1}{3},\ f_{m2} = 25\text{ kHz}$

$$s(t) = A_c\cos 2\pi f_c t + \frac{A_c\mu_1}{2}\cos 2\pi(f_c - f_{m1})t + \frac{A_c\mu_1}{2}\cos 2\pi(f_c + f_{m1})t$$

$$-\frac{A_c\mu_2}{2}\cos 2\pi(f_c - f_{m2})t - \frac{A_c\mu_2}{2}\cos 2\pi(f_c + f_{m2})t$$

The spectrum of $s(t)$ is thus obtained as below,



(ii) Modulation index of multi-tone AM signal μ_T is given as

$$\mu_T = \sqrt{\mu_1^2+\mu_2^2} = \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{9}} = \frac{\sqrt{13}}{6} = 0.6$$

(iii) The output of HPF will contain the SSB-USB components only, given as

$$Y(f) = 3[\delta(f-f_c)+\delta(f+f_c)]+\frac{3}{4}[\delta(f-612.5\text{ k})+\delta(f+612.5\text{ k})$$

$$-\frac{1}{2}[\delta(f-625\,k)+\delta(f+625k)]$$

Apply IFT,

$$y(t) = 6\cos(2\pi\times 600\times 10^3 t)+\frac{3}{2}\cos(2\pi\times 612.5\times 10^3 t)$$

$$-\cos[2\pi\times 625\times 10^3 t]$$

With respect to a 600 kHz frequency, the time domain complex envelope of $y(t)$ is given by

$$\hat{y}(t) = \frac{3}{2}e^{(j25\pi t)}-e^{(j50\pi t)}$$

Quadrature component of filter output is therefore calculated as

$$y_s(t) = \text{Im}(\hat{y}(t)) = \frac{3}{2}\sin(25\pi t)-\sin(50\pi t)$$

Note: Alternatively, the quadrature component $y_s(t)$ can be found by multiplying the signal by $-2\sin(\omega_c t)$ and passing it through a low-pass filter. We have,

$$y'(t) = -12\cos(2\pi\times 600\times 10^3 t)\sin(2\pi\times 600\times 10^3 t) - 3\cos(2\pi\times 612.5\times 10^3 t)$$

$$\sin(2\pi\times 600\times 10^3 t) + 2\cos(2\pi\times 625\times 10^3 t)\sin(2\pi\times 600\times 10^3 t)$$

$$y'(t) = -6[\sin(2400\pi\times 10^3 t) - \sin(2\pi\times 10^3 t)] - 1.5[\sin(2\pi\times 1212.5\times 10^3 t)$$

$$-\sin(2\pi\times 12.5\times 10^3 t)] + [\sin(2\pi\times 1225\times 10^3 t) - \sin(2\pi\times 25\times 10^3 t)]$$

After passing through the low pass filter, we get the quadrature component $y_s(t)$ as

$$y_s(t) = \frac{3}{2}\sin(25\pi\times10^3 t)] - \sin(50\pi\times10^3 t)$$

Q.7 (a) Solution:

(i) The input R is at station A and so the input at station B is made zero. Let the output is C_1. Since there is no input at station B, that summing point can be removed and the resultant block diagram will be as shown in figure (a).

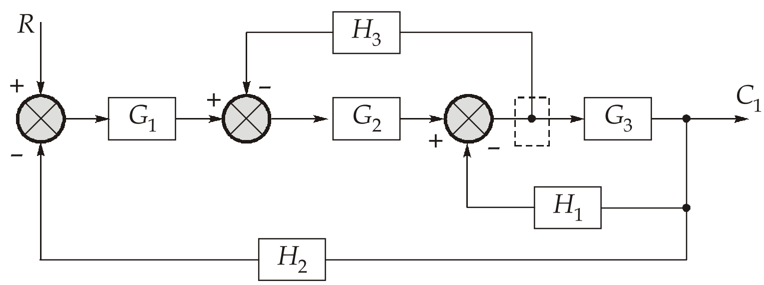


Figure (a)

Step 1: Moving the take-off point in figure (a) after block G_3, the block diagram will be as shown in figure (b).

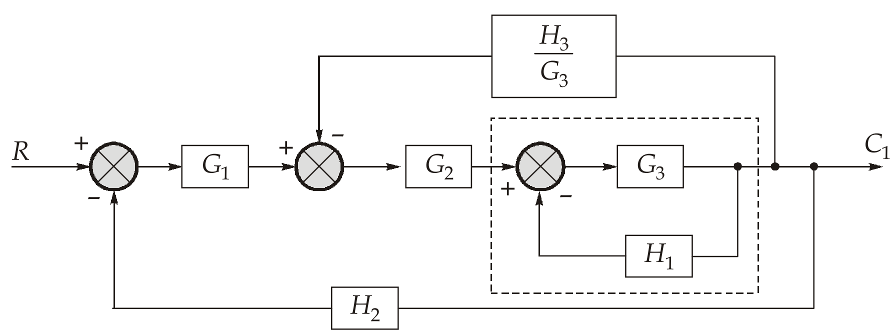


Figure (b)

Step 2: Eliminating the inner loop in figure (b) and combining the result with G_2 and rearranging the block diagram will be as shown in figure (c).

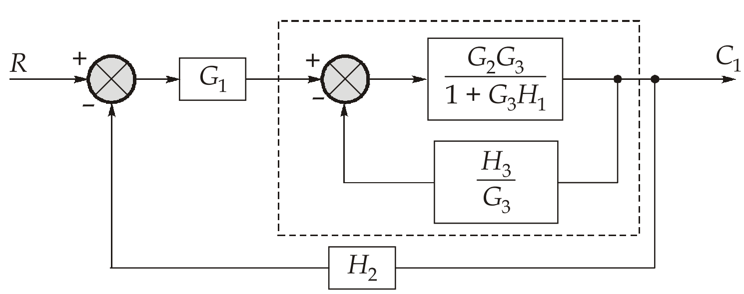


Figure (c)

Step 3: Eliminating the inner loop in figure (c), the block diagram is as shown in figure (d).

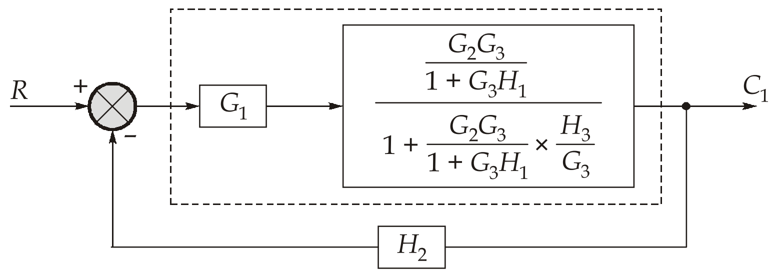


Figure (d)

Step 4: Combining the blocks in cascade in figure (d), the clock diagram will be as shown in figure (e).

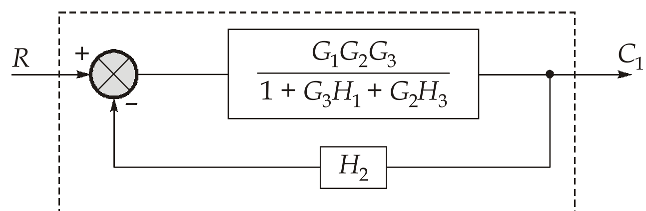


Figure (e)

Step 5: Eliminating the only loop in figure (e), the block diagram will be as shown in figure (f).

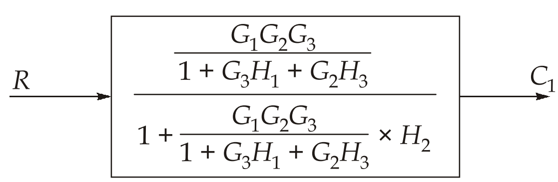


Figure (f)

Step 6: After simplification, the closed-loop transfer function is

$$\frac{C_1}{R} = \frac{G_1G_2G_3}{1+G_3H_1+G_2H_3+G_1G_2G_3H_2}$$

(ii) The input R is at station B. So the input at station A is made zero. Let the output be C_2. Since the input at station A is zero, the corresponding summing point can be removed and a negative sign can be attached to the feedback path gain H_2. The resulting block diagram is shown in figure (g).

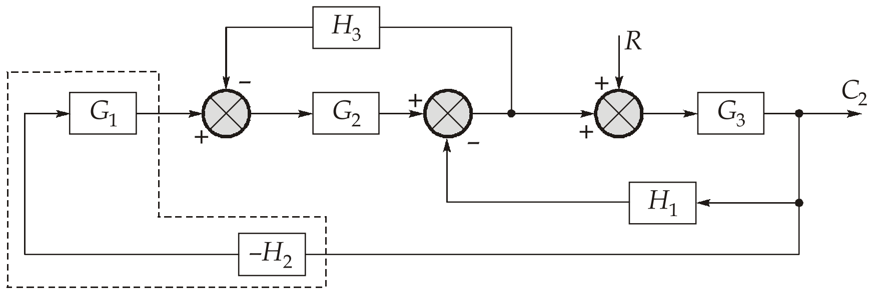


Figure (g)

Step 1: Combining the blocks G_1 and $-H_2$ in cascade into a single block and rearranging the diagram in figure (g), the resultant block diagram is as shown in figure (h).

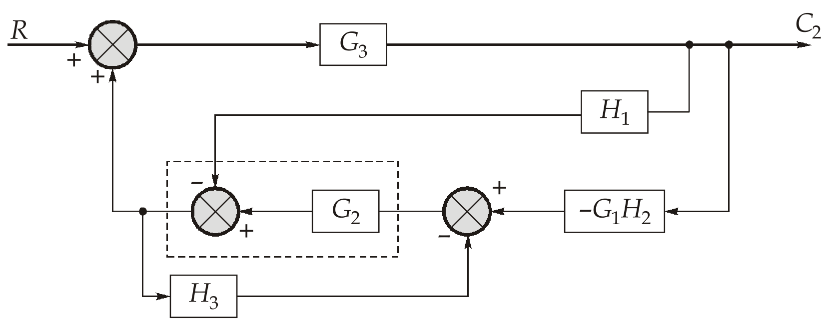


Figure (h)

Step 2: Moving the summing point before the block G_2 in figure (h), the block diagram will be as shown in figure (i).

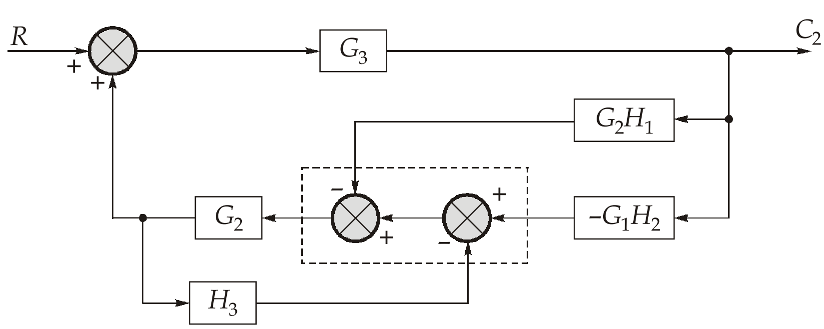


Figure (i)

Step 3: Interchanging the summing points in figure (i), the block diagram will be as shown in figure (j).

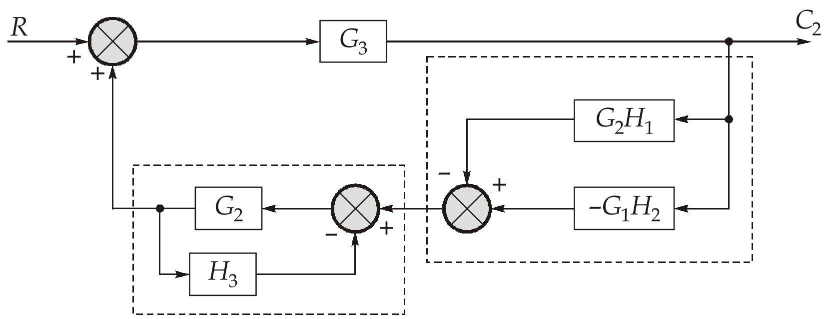


Figure (j)

Step 4: Eliminating the loop and combining the parallel blocks in the feedback path in figure (j), the block diagram will be as shown in figure (k).

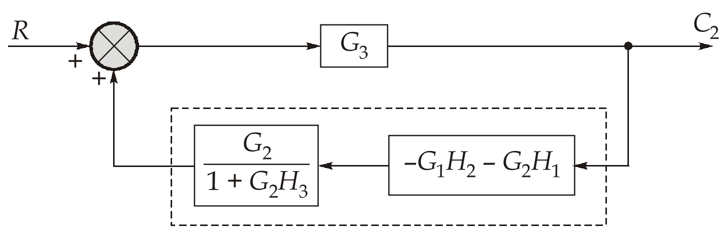


Figure (k)

Step 5: Combining the two blocks in cascade in figure (k), the block diagram will be as shown in figure (l).

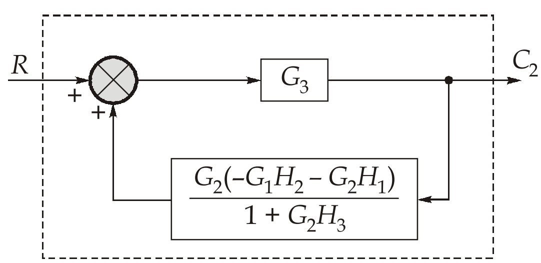


Figure (l)

Step 6: Eliminating the loop in figure (l), the transfer function will be as follows:

$$\frac{C_2}{R} = \frac{G_3}{1+\dfrac{G_2(G_1H_2+G_2H_1)}{1+G_2H_3}}$$

$$= \frac{G_3(1+G_2H_3)}{1+G_2H_3+G_2G_1H_2+G_2^2H_1}$$

Q.7 (b) Solution:

(i) **1.** Since,

$$(sI - A) = \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}$$

We obtain

$$(sI - A)^{-1} = \begin{bmatrix} 1 & \dfrac{1}{s(s+2)} \\ 0 & \dfrac{1}{s+2} \end{bmatrix}$$

Hence,

$$e^{At} = L^{-1}[(sI - A)^{-1}] = \begin{bmatrix} 1 & \dfrac{1}{2}\left(1-e^{-2t}\right) \\ 0 & e^{-2t} \end{bmatrix}$$

$$x(t) = e^{At}\,x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

2.

$$x(t) = e^{At}\,x(0) + \int_0^t e^{A(t-\tau)}B\,u(\tau)\,d\tau$$

Now,

$$\int_0^t e^{A(t-\tau)}B\,u(\tau)\,d\tau = \begin{bmatrix} \dfrac{1}{2}\displaystyle\int_0^t \left\{1-e^{-2(t-\tau)}\right\}d\tau \\ \displaystyle\int_0^t e^{-2(t-\tau)}d\tau \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{1}{4}+\dfrac{1}{2}t+\dfrac{1}{4}e^{-2t} \\ \dfrac{1}{2}\left(1-e^{2t}\right) \end{bmatrix}$$

Therefore,

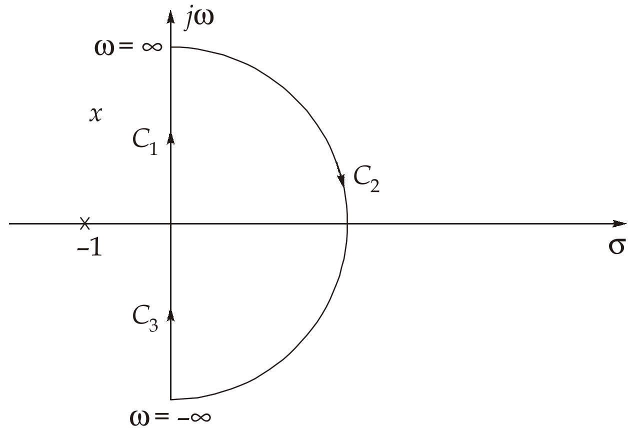
$$x_1(t) = \frac{3}{4}+\frac{1}{2}t+\frac{1}{4}e^{-2t}$$

$$x_2(t) = \frac{1}{2}\left(1-e^{-2t}\right)$$

(ii) The open loop transfer function is given by

$$G(s) = \frac{8k}{(s+1)\left(s^2+2s+2\right)}$$

The Nyquist contour is as shown below,



$$p = 0$$

Substituting $s = jw$, we get

$$G(j\omega) = \frac{8k}{(j\omega+1)(2-\omega^2+2j\omega)} = \frac{8k\left[2-3\omega^2-j(4\omega-\omega^3)\right]}{(2-3\omega^2)^2+(4\omega-\omega^3)^2}$$

Nyquist plot in $G(s)\,H(s)$ plane is

Corresponding to $C_1 : s = j\omega \quad \omega : 0 \to \infty$

$$G(j\omega) = \frac{8k}{(j\omega+1)(2-\omega^2+2j\omega)}$$

$$M = \frac{8k}{\sqrt{\omega^2+1}\sqrt{(2-\omega^2)^2+4\omega^2}}$$

$$\phi = -\tan^{-1}\omega - \tan^{-1}\frac{2\omega}{2-\omega^2}$$

	M	$\angle\phi$
$\omega = 0^-$	$4K$	0°
$\omega = \infty$	0	90°

Corresponding to $C_3 : s = -j\omega \quad \omega : \infty \to 0$

$$G(-j\omega)H(-j\omega) = \frac{8k}{(1-j\omega)(2-\omega^2+2j\omega)}$$

$$M = \frac{8k}{\sqrt{1+\omega^2}\sqrt{(2-\omega^2)^2+4\omega^2}}$$

$$\phi = -\tan^{-1}(-\omega) - \tan^{-1}\left(\frac{-2\omega}{2-\omega^2}\right)$$

$$= \tan^{-1}(\omega)+\tan^{-1}\left(\frac{2\omega}{2-\omega^2}\right)$$

	M	$\angle\phi$
$\omega = 0^-$	$4K$	$0°$
$\omega = \infty$	0	$90°$

Corresponding to C_2 :

$$G(s)\,H(s) = \lim_{R\to\infty} Re^{j\theta} \qquad \theta = 90° \text{ to } -90°$$

So,
$$G(s)\,H(s) = \lim_{R\to\infty} \frac{8k}{(Re^{j\theta}+1)\left[(Re^{j\theta})^2+Re^{j\theta}+2\right]}$$

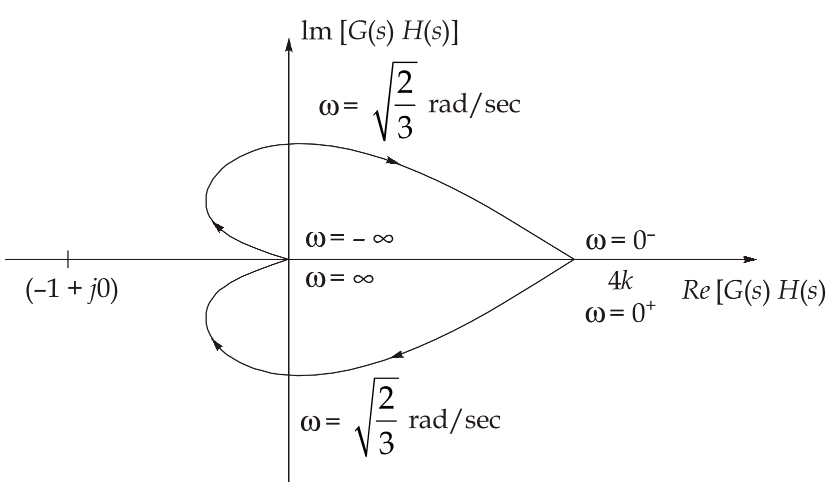
$$= \lim_{R\to\infty}\frac{8k}{Re^{j\theta}(R^2e^{j2\theta})}$$

$$= \lim_{R\to\infty}\frac{8k}{R^3e^{3j\theta}}$$

$$= \lim_{R\to\infty}\frac{8k}{R^3}e^{j(180-3\theta)}$$

Thus $\quad M = 0 : \phi = 180° - 3\theta = -90°$ to $90°$

The Nyquist plot in $G(s)H(s)$ plane:



The Nyquist plot will intersect with 90° line where $Re\{G(j\omega)\,H(j\omega)\} = 0$

$$2 - 3\omega^2 = 0$$

$$\omega = \pm\sqrt{\frac{2}{3}}$$

Number of encirclements of the point (–1, $j0$) by the Nyquist plot : $N = 0$

$$P = 0$$

Nyquist stability criterion

According to the Nyquist stability criterion.

$$N = P - Z$$

$$0 = 0 - Z$$

So, $$Z = 0$$

Hence, no pole of the closed loop system lies in the right half of s-plane. Therefore, the closed loop system is stable.

Q.7 (c) Solution:

(i) 1.
$$SQNR = \frac{P_S}{P_N}$$

Where, $P_S \rightarrow$ Signal power, $P_N \rightarrow$ Quantization Noise power

For PCM, quantization noise power,

$$P_N = \frac{\Delta^2}{12} = \left(\frac{2A}{L}\right)^2 \times \frac{1}{12} = \left(\frac{2\times 2}{L}\right)^2 \times \frac{1}{12}$$

$$= \left(\frac{4}{32}\right)^2 \times \frac{1}{12}$$

$$\therefore \quad P_N = \frac{1}{768}\text{W}$$

Signal power, $$P_S = E[x^2(t)] = \int_{-2}^{0} x^2 f_X(x)dx + \int_{0}^{2} x^2 f_X(x)dx$$

$$= \int_{-2}^{0} x^2\left(\frac{x+2}{4}\right)dx + \int_{0}^{2} x^2\left(\frac{-x+2}{4}\right)dx$$

$$= \frac{1}{4}\left[\frac{x^4}{4} + \frac{2x^3}{3}\right]_{-2}^{0} + \frac{1}{4}\left[\frac{-x^4}{4} + \frac{2x^3}{3}\right]_{0}^{2}$$

$$= \frac{1}{4}\left[\frac{-16}{4} + \frac{2 \times 8}{3}\right] + \frac{1}{4}\left[\frac{-16}{4} + \frac{2 \times 8}{3}\right] = \frac{1}{3} + \frac{1}{3}$$

$\therefore$ $$P_S = \frac{2}{3}\text{W}$$

$$\text{SQNR} = \frac{P_S}{P_N} = \frac{2/3}{1/768} = 512$$

$$[\text{SQNR}]_{\text{dB}} = 10 \log_{10} 512$$

$$[\text{SQNR}] = 27.093 \text{ dB}$$

2. Bit rate, $R_b = nf_s$ ...(i)

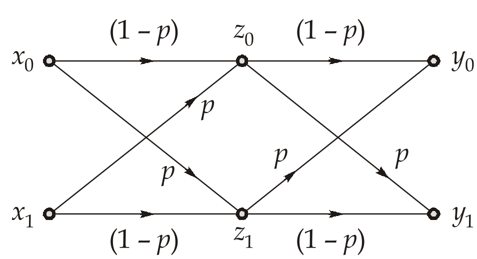
Given, $L = 32 = 2^n$, $f_m = 5$ kHz

$\therefore$ $n = 5$

$R_b = 5 \times 2 \times 5 \times 10^3$ ($\because f_s = 2f_m$ at Nyquist Rate)

$= 50$ Kbps

(ii) Assuming p as the channel transition probability, the channel transition diagram can be drawn as below:



The overall channel matrix,

$$P(Y|X) = [P(Z|X)] \cdot [P(Y|Z)]$$

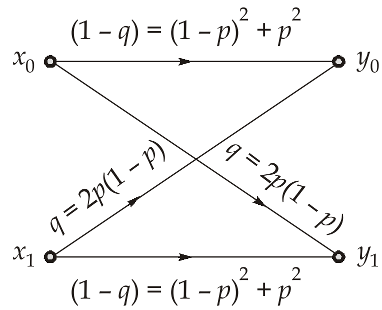
$$P(Z|X) = \begin{bmatrix} (1-p) & p \\ p & (1-p) \end{bmatrix},$$

$$P(Y|Z) = \begin{bmatrix} (1-p) & p \\ p & (1-p) \end{bmatrix}$$

$$P(Y|X) = \begin{bmatrix} (1-p) & p \\ p & (1-p) \end{bmatrix}\begin{bmatrix} (1-p) & p \\ p & (1-p) \end{bmatrix}$$

$$P(Y|X) = \begin{bmatrix} (1-p)^2 + p^2 & 2p(1-p) \\ 2p(1-p) & (1-p)^2 + p^2 \end{bmatrix}$$

Equivalent single binary symmetric channel can be shown as



The capacity of cascaded connection,

$$C = 1 + q \log_2 q + (1-q)\log_2(1-q)$$

$$C = 1 + 2p(1-p)\log_2(2p(1-p)) + ((1-p)^2 + p^2)$$
$$\log_2((1-p^2) + p^2)$$

If,
$$p = 0.4$$
$$q = 2p(1-p) = 0.48$$
$$1 - q = 1 - 0.48 = 0.52$$
$$C = 1 + 0.48\log_2(0.48) + 0.52\log_2(0.52)$$
$$= 1 - 0.99884$$
$$C = 0.00116 \text{ bits/channel}$$

Q.8 (a) Solution:

(i) On taking Laplace transform of $\dot{x} = Ax + Bu$, we get

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$(sI - A)X(s) = x(0) + BU(s)$$
$$X(s) = (sI - A)^{-1}[x(0) + BU(s)] \quad \text{...(i)}$$

Similarly on taking Laplace transform of $y = Cx + Du$, we get

$$Y(s) = CX(s) + DU(s) \quad \text{...(ii)}$$

On substituting the value of $X(s)$ from equation (i), we get

$$Y(s) = C((sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)) + DU(s)$$

Now, to get the desired transfer function,

$$T(s) = C(sI - A)^{-1}B + D$$

We have to assume that the system has zero initial condition, i.e.,

$$x(0) = 0$$

We get
$$Y(s) = C(sI - A)^{-1}BU(s) + DU(s)$$

Therefore, $\frac{Y(s)}{U(s)} = [C(sI - A)^{-1} B + D]$

(ii) We have,

Resonant frequency, $\omega_r = 12$ rad/sec

$$\omega_r = \omega_n\sqrt{1-2\xi^2} = 12 \quad ...(i)$$

and magnitude resonant frequency,

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = 1.15$$

$$(2\xi\sqrt{1-\xi^2})(1.15) = 1$$

$$4\xi^2(1 - \xi^2)(1.32) = 1$$

$$(4x^2 - 4x^4)(1.32) = 1$$

$$5.28\xi^4 - 5.28\xi^2 + 1 = 0$$

$$\xi^2 = 0.75; \quad 0.25$$

$$\xi = 0.87 \text{ and } 0.5$$

For maximum peak magnitude $\xi < 0.7$. Therefore

$$\xi = 0.5$$

Now, from equation (i)

$$12 = \omega_n\sqrt{1-2(0.5)^2}$$

$$\omega_n = \frac{12}{\sqrt{1-2(0.25)}}$$

$$\omega_n = 12\sqrt{2} \text{ rad/sec}$$

As, we get $\omega_n = 12\sqrt{2}$ rad/sec and $\xi = 0.5$

Then, from open loop transfer function, we get

$$\omega_n^2 = k$$

$$k = (12\sqrt{2})^2$$

$$k = 288$$

and $2\xi\omega_n = a$

$$2\times0.5\times12\sqrt{2} = a$$

$$a = 12\sqrt{2}$$

- For 2% tolerance, the settling time,

$$t_s = \frac{4}{\xi\omega_n}$$

$$t_s = \frac{4}{0.5\times 12\sqrt{2}}$$

$$t_s = 0.47 \text{ sec}$$

- $$\text{Bandwidth, BW} = \omega_n\sqrt{\left[(1-2\xi^2)+\sqrt{2-4\xi^2+4\xi^4}\right]}$$

$$\text{BW} = 12\sqrt{2}\times\sqrt{(1-2\times(0.5)^2)+\sqrt{2-4\times(0.5)^2+4\times(0.5)^4}}$$

$$\text{BW} = 21.59 \text{ rad/sec}$$

Thus, we get

1. $k = 288$
2. $a = 12\sqrt{2}$
3. Settling time, $t_s = 0.47$ sec
4. Bandwidth, BW = 21.59 rad/sec

Q.8 (b) Solution:

We know that,

Due to presence of a pole slope of the curve decrease by 20 dB/dec whereas because a zero slope of the curve is improve by 20 dB/dec.

From the curve, we get

At,

$s = 0$, ω and ω_3 poles are present whereas

at $s = \omega_2$ one zero is present.

Thus we can write transfer function as,

$$G(s) = \frac{K(s+\omega_2)}{s(s+\omega_1)(s+\omega_3)}$$

Now, our task is to calculate, K, ω_2, ω_2 and ω_3.

A pole is at origin i.e.,

$$20 \log K - 20 \log \omega = 36 \text{ at } \omega = \omega_1$$

$$20 \log K - 20 \log \omega_1 = 36 \quad \text{...(i)}$$

Now using slope concept in between ω_1 and 4 rad/sec, we get,

$$\frac{y_2 - y_1}{x_2 - x_1} = -40 \text{ dB/dec}$$

$$\frac{36-0}{\log \omega_1 - \log 4} = -40 \text{ dB/dec}$$

$$36 = -40(\log \omega_1 - \log 4)$$

$$36 = -40 \log \omega_1 + 40 \log 4$$

$$\frac{36 - 40 \log 4}{-40} = \log \omega_1$$

$$\omega_1 = 0.5 \text{ rad/sec}$$

Substituting $\omega_1 = 0.5$ rad/sec in equation (i) we get

$$20 \log K - 20 \log 0.5 = 36$$

$$20 \log K = 36 + 20 \log 0.5$$

$$K = 31.55$$

Similarly, using slope concept 4 rad/sec and ω_2, we get

$$\frac{y_2 - y_1}{x_2 - x_1} = -40 \text{ dB/dec}$$

$$\frac{0-(-12)}{\log 4 - \log \omega_2} = -40$$

$$\frac{12}{\log 4 - \log \omega_2} = -40$$

$$12 = -40 \log 4 + 40 \log \omega_2$$

$$\frac{12 + 40 \log 4}{40} = \log \omega_2$$

$$\omega_2 = 7.98 \text{ rad/sec}$$

Now, on applying slope concept in between $\omega_2 = 7.98$ rad/sec and ω_3, we get

$$\frac{y_2 - y_1}{x_2 - x_1} = -20 \text{ dB/dec}$$

$$\frac{-12-(-21)}{\log \omega_2 - \log \omega_3} = -20$$

$$\frac{-12+21}{\log 7.98 - \log \omega_3} = -20$$

$$9 = -20 \log 7.98 + 20 \log \omega_3$$

$$\frac{9 + 20\log 7.98}{20} = \log \omega_3$$

$$\omega_3 = 22.49 \text{ rad/sec}$$

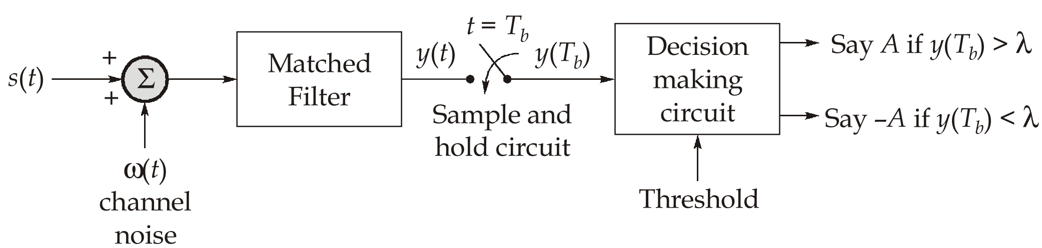
On combining all the term, we get the open-loop transfer as,

$$G(s) = \frac{31.55\left(\frac{s}{7.98}+1\right)}{s\left(\frac{s}{0.5}+1\right)\left(\frac{s}{22.49}+1\right)}$$

$$G(s) = \frac{31.55(0.125s+1)}{s(2s+1)(0.044s+1)}$$

Q.8 (c) Solution:

The given receiver circuit can be represented as below:



Here, $s(t)$ is a digital NRZ signal defined as

$$s(t) = \begin{cases} +A \text{ when binary '1' is transmitted} \\ -A, \text{ when binary '0' transmitted} \end{cases}$$

Let T_b be the bit duration of NRZ signal.

The digital NRZ signal when passes through the channel, it introduces white Gaussian Noise $\omega(t)$ having PSD of $\frac{N_0}{2}$. Expressing $s(t)$ in terms of orthonormal basis functions, we can write

$$s_1(t) = A\sqrt{T_b}\,\phi(t), \text{ when '}+A\text{' is transmitted}$$

$$s_2(t) = -A\sqrt{T_b}\,\phi(t), \text{ when '}-A\text{' is transmitted}$$

Here, $\phi(t)$ represents a unit energy rectangular pulse of duration T_b.

(i) Output of the sample and hold circuit when 'A' is transmitted.

When 'A' is transmitted, $s(t) = A\sqrt{T_b}\,\phi(t)$

The received signal $x(t)$ at the receiver can be written as

$$x(t) = A\sqrt{T_b}\,\phi(t)+\omega(t),\ 0 \le t \le T_b$$

The matched filter output is given by

$$y(t) = x(t) * h(t)$$

where $h(t)$ is the impulse response of the matched filter.

We can write, $$y(t) = \int_0^t x(\tau)\cdot h(t-\tau)\cdot d\tau$$

The sample and hold circuit sample the output of the filter at $t = T_b$.

We obtain, $$y(T_b) = \int_0^{T_b} x(\tau)\cdot h(T_b-\tau)\cdot d\tau$$

For matched filter,

$$h(T_b - \tau) = x(\tau)A\sqrt{T_b}\,\phi(\tau)$$

Therefore, $$y(T_b) = \int_0^{T_b}\left[A\sqrt{T_b}\,\phi(\tau)+\omega(\tau)\right]\cdot A\sqrt{T_b}\phi(\tau)\cdot d\tau$$

$$y(T_b) = A^2T_b\int_0^{T_b}\phi^2(\tau)d\tau + A\sqrt{T_b}\int_0^{T_b}\omega(\tau)\phi(\tau)\,d\tau$$

Since, $\phi(\tau)$ represents a unit energy pulse. Hence,

$$\int_0^{T_b}\phi^2(\tau)d\tau = 1$$

Therefore, $$y(T_b) = A^2T_b + y_n(T_b)$$

Where $y_n(T_b)$ represents the noise component.

(ii) Variance of the noisy signal at the output of sample and hold circuit.

The noise component at the output of sample and hold circuit is

$$y_n(T_b) = \int_0^{T_b}\omega(\tau)h(T_b-\tau)d\tau$$

Since, $E[y_n(T_b)] = 0$. The variance of the noise signal can be calculated as

$$\sigma_n^2 = E\left[y_n^2(T_b)\right] = \text{PSD}[\omega(\tau)] \times \text{Energy}[h(T_b - \tau)]$$

$$\sigma_n^2 = \frac{N_0}{2} \times A^2 T_b$$

(iii) Probability of error:

We have,
$$y(T_b) = \begin{cases} -A^2T_b + y_n(T_b), \text{ when '}-A\text{' is transmitted} \\ A^2T_b + y_n(T_b), \text{ when '}+A\text{' is transmitted} \end{cases}$$

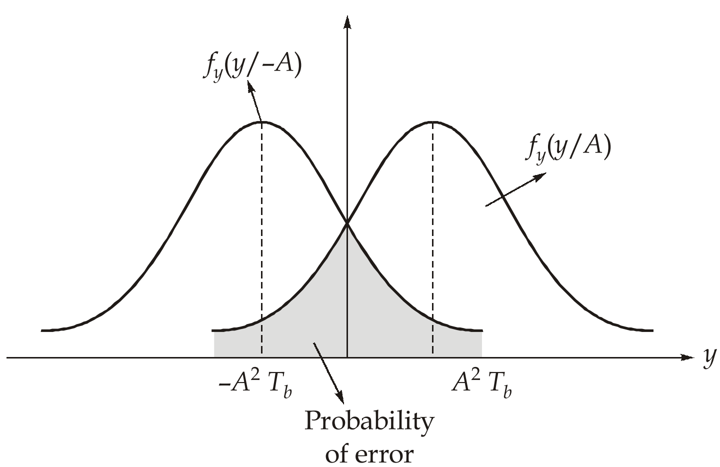
when $y_n(T_b)$ has zero mean and variance σ_n^2.

The probability density function of $y(T_b)$ when '$-A$' is transmitted is given by

$$f_y(y/-A) = \frac{1}{\sqrt{\pi N_0 A^2 T_b}} e^{-\frac{(y+A^2T_b)^2}{N_0 A^2 T_b}}$$

When $+A$ is transmitted,

$$f_y(y/+A) = \frac{1}{\sqrt{\pi N_0 A^2 T_b}} e^{\left[-\frac{(y-A^2T_b)^2}{N_0 A^2 T_b}\right]}$$



The two probability density functions, $f_y(y/-A)$ and $f_y(y/+A)$ intersect at $y = 0$. For the maximum likelihood algorithm/technique, the threshold of the decision making circuit is '0'.

$$P_e = P(A) \times \int_{-\infty}^{0} f_y(y/A)dy + P(-A)\int_{0}^{\infty} f_y(y/-A)dy$$

Assuming equiprobable symbols,

$$P(A) = P(-A) = \frac{1}{2}$$

$$P_e = \frac{1}{2}\left[\int_{-\infty}^{0} \frac{1}{\sqrt{\pi N_0 A^2 T_b}} e^{-\left[\frac{(y-A^2T_b)^2}{N_0A^2T_b}\right]} \cdot dy\right]$$

$$+\frac{1}{2}\left[\int_{0}^{\infty} \frac{1}{\sqrt{\pi N_0 A^2 T_b}} e^{-\left[\frac{(y+A^2T_b)^2}{N_0A^2T_b}\right]} \cdot dy\right]$$

Let $u = \frac{y - A^2T_b}{\sqrt{N_0A^2T_b}}$ and $v = \frac{y + A^2T_b}{\sqrt{N_0A^2T_b}}$, we get

$$P_e = \frac{1}{2}\left[\frac{1}{\pi}\int_{-\infty}^{-\frac{A\sqrt{T_b}}{\sqrt{N_0}}} e^{-u^2} du + \frac{1}{\pi}\int_{\frac{A\sqrt{T_b}}{\sqrt{N_0}}}^{\infty} e^{-v^2} dv\right]$$

Let $u = -z$

$$P_e = \frac{1}{2}\left[\frac{1}{\sqrt{\pi}}\int_{\frac{A\sqrt{T_b}}{\sqrt{N_0}}}^{\infty} e^{(-z^2)} dz + \frac{1}{\sqrt{\pi}}\int_{\frac{A\sqrt{T_b}}{\sqrt{N_0}}}^{\infty} e^{(-v^2)} dv\right]$$

$$P_e = \frac{1}{\sqrt{\pi}}\int_{\frac{A\sqrt{T_b}}{\sqrt{N_0}}}^{\infty} e^{(-v^2)} dv = \frac{1}{2} erfc\left[\frac{A\sqrt{T_b}}{\sqrt{N_0}}\right]$$

Since, $Q(x) = \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}}\right)$. We can write

$$P_e = Q\left(\sqrt{\frac{2A^2T_b}{N_0}}\right)$$

❍❍❍❍